



Construction of Actuarial Models

Fourth Edition

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Solutions to practice questions – Chapter 1

Solution 1.1

$$M_X(t) = (1 - 10t)^{-2} \Rightarrow M'_X(t) = 20(1 - 10t)^{-3}, M''_X(t) = 600(1 - 10t)^{-4}$$

$$E[X] = M'_X(0) = 20, E[X^2] = M''_X(0) = 600, \text{var}(X) = 600 - 20^2 = 200$$

Solution 1.2

$$M_N(t) = E[e^{tN}] = \sum_{k=0}^4 e^{tk} \Pr(N=k) = \sum_{k=0}^4 e^{tk} \binom{4}{k} 0.7^k 0.3^{4-k}$$

$$= \sum_{k=0}^4 \binom{4}{k} (0.7e^t)^k 0.3^{4-k} = (0.7e^t + 0.3)^4$$

binomial theorem: $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$

$$P_N(t) = M_N(\ln(t)) = (0.7e^{\ln(t)} + 0.3)^4 = (0.7t + 0.3)^4$$

Solution 1.3

$$P_{N_1+N_2}(t) = E[t^{N_1+N_2}] = E[t^{N_1}]E[t^{N_2}] = P_{N_1}(t)P_{N_2}(t) = (0.7t + 0.3)^8$$

Solution 1.4

The possible values of $X_1 + X_2 + X_3 + X_4$ are: 4, 5, 6, 7, 8. Here are the first few probabilities:

$$f_X^{*4}(4) = f_X^{*3}(3)f_X(1) = 0.2401$$

$$f_X^{*4}(5) = f_X^{*3}(3)f_X(2) + f_X^{*3}(4)f_X(1) = 0.4116$$

We leave it to the reader to check the following:

$$f_X^{*4}(6) = 0.2646, f_X^{*4}(7) = 0.0756, f_X^{*4}(8) = 0.0081$$

Solution 1.5

We have $S = X_1 + \dots + X_{100}$ where the various X_i are independent and identically distributed like X :

$$f_X(x) = \begin{cases} 0.80 & x = 0 \text{ (discrete part)} \\ 0.20/1900 & 100 \leq x \leq 2000 \text{ (continuous part)} \end{cases}$$

$$E[X^k] = 0.8 \times 0^k + 0.2 \int_{100}^{2,000} x^k \frac{1}{1,900} dx = 0.2 \left(\frac{2,000^{k+1} - 100^{k+1}}{1,900(k+1)} \right) \Rightarrow$$

$$E[X] = 210, \quad E[X^2] = 280,666.66, \quad \text{var}(X) = 236,566.67$$

Now move up to the aggregate level:

$$E[S] = 100 E[X] = 21,000$$

$$\text{var}(S) = 100 \text{var}(X) = 23,656,666.67$$

$$95\text{-th percentile} = E[S] + 1.645 \sqrt{\text{var}(S)} = 29,001$$

Solution 1.6

Let Y be the height and let I be an indicator for the sex of an individual:

$$I = \begin{cases} 0 & \text{male} \\ 1 & \text{female} \end{cases}, \quad \text{where } \Pr(I=0) = 0.45, \Pr(I=1) = 0.55$$

The given data can be interpreted as being conditional means and variances for the two groups:

$I = i$	$\Pr(I = i)$	$E[Y I = i]$	$\text{var}(Y I = i)$
0	0.45	70	$4^2 = 16$
1	0.55	67	$3^2 = 9$

Now apply the double expectation theorem:

$$E[Y] = E[E[Y|I]] = 0.45 \times 70 + 0.55 \times 67 = 68.35$$

$$E[\text{var}(Y|I)] = 0.45 \times 16 + 0.55 \times 9 = 12.15$$

$$\begin{aligned} \text{var}(E[Y|I]) &= E[(E[Y|I])^2] - (E[E[Y|I]])^2 \\ &= 0.45 \times 70^2 + 0.55 \times 67^2 - 68.35^2 = 2.22750 \end{aligned}$$

$$\text{var}(Y) = 12.15 + 2.22750 = 14.3775$$

So the standard deviation is 3.7918.

Solution 1.7

IRM: $X_1 = 140 + 30 = 170$, $X_2 = 57 + 90 = 147$, $X_3 = 0$, $X_4 = 100$
 $S = X_1 + X_2 + X_3 + X_4 = 417$

CRM: $N = 5$, $Y_1 = 57$, $Y_2 = 100$, $Y_3 = 90$, $Y_4 = 140$, $Y_5 = 30$
 $S = Y_1 + \dots + Y_N = Y_1 + \dots + Y_5 = 417$

Solution 1.8

$$E[N] = \text{var}(N) = \lambda = 5 \quad , \quad E[Y] = \theta = 1/0.002 = 500 \quad , \quad \text{var}(Y) = \theta^2 = 250,000$$

$$E[S] = E[N] E[Y] = 5 \times 500 = 2,500$$

$$\text{var}(S) = E[N] \text{var}(Y) + (E[Y])^2 \text{var}(N) = 2,500,000$$

$$M_N(t) = e^{\lambda(e^t - 1)} = e^{5(e^t - 1)} \quad , \quad M_Y(t) = (1 - \theta t)^{-1} = (1 - 500t)^{-1}$$

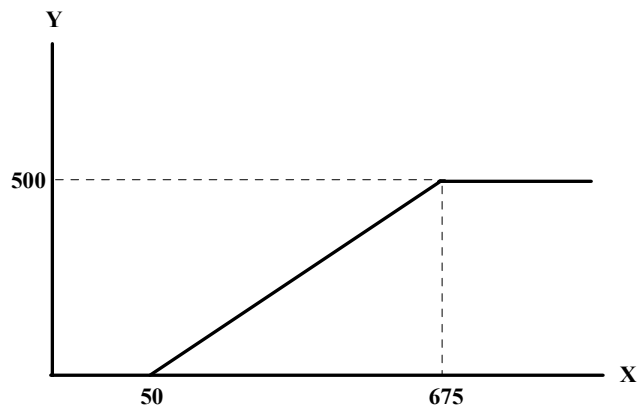
$$\Rightarrow M_S(t) = M_N(\ln(M_Y(t))) = e^{5(M_Y(t) - 1)} = e^{5((1 - 500t)^{-1} - 1)} \quad \text{for } t < 1/500$$

$$\Pr(S = 0) = \Pr(N = 0) = e^{-5} \quad \text{since } Y \text{ is continuous}$$

Solution 1.9

$$Y = \begin{cases} 0 & \text{if } X \leq 50 \\ 0.8(X - 50) & \text{if } 50 < X \leq 675 \\ 500 & \text{if } 675 < X \end{cases}$$

since $500 = 0.8(X - 50)$ when $X = 675$.



Solution 1.10

From the formula in Solution 1.9, we have:

$$F_Y(y) = \Pr(Y \leq y) = 0 \quad \text{when } y < 0$$

For y in the range $[0, 500)$, we have:

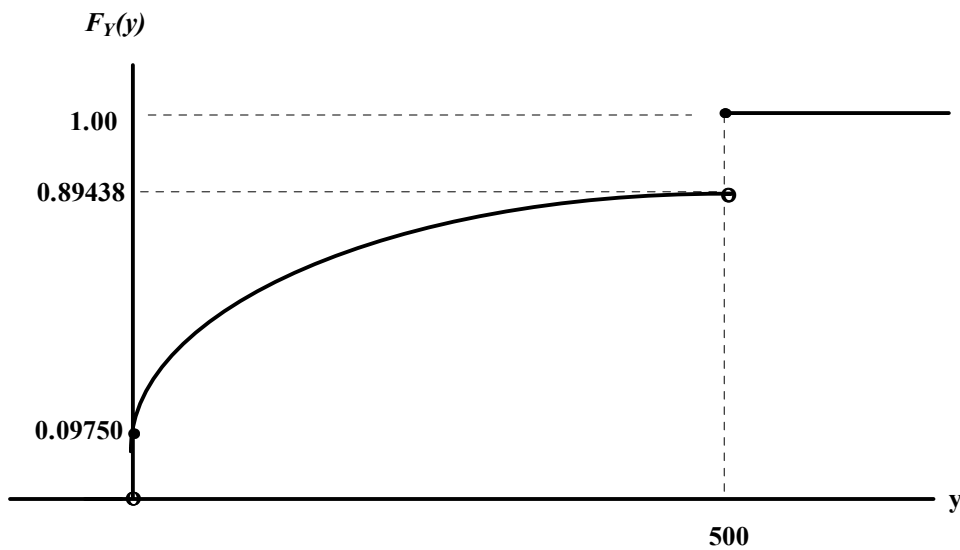
$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) = \Pr(0.8(X - 50) \leq y) = \Pr(X \leq 1.25y + 50) \\ &= \int_0^{1.25y+50} \frac{2(1,000-x)}{1,000^2} dx = 1 - \left(\frac{760-y}{800}\right)^2 \end{aligned}$$

For $y \geq 500$ we have:

$$F_Y(y) = 1$$

since the largest possible value of Y is 500.

Here is the graph:

**Solution 1.11**

The discrete part is concentrated at the jump discontinuities:

$$f_Y(y) = \begin{cases} 1 - \left(\frac{760}{800}\right)^2 = 0.09750 & \text{when } y = 0 \quad (\text{discrete part}) \\ \left(\frac{260}{800}\right)^2 = 0.10563 & \text{when } y = 500 \quad (\text{discrete part}) \\ F'(y) = \frac{2(760-y)}{800^2} & \text{when } 0 < y < 500 \quad (\text{continuous part}) \end{cases}$$

Solution 1.12

$$\begin{aligned}
 E[Y] &= 0 \times (0.09750) + 500 \times (0.10563) + \int_0^{500} y \times \frac{2(760-y)}{800^2} dy \\
 &= 52.81250 + \left(\frac{(760y^2 - 2y^3/3)}{800^2} \right) \Bigg|_0^{500} = 52.81250 + 166.67 = 219.48
 \end{aligned}$$

Solution 1.13

$$\begin{aligned}
 E[Y] &= \int_0^{50} 0 \times f_X(x) dx + \int_{50}^{675} 0.8(x-50) f_X(x) dx + \int_{675}^{1,000} 500 f_X(x) dx \\
 &= \int_{50}^{675} 0.8(x-50) \times \frac{2(1,000-x)}{1,000^2} dx + \int_{675}^{1,000} 500 \times \frac{2(1,000-x)}{1,000^2} dx \\
 &= \int_0^{500} u \times \frac{2(760-u)}{800^2} du + 500 \times \left(\frac{325}{1,000} \right)^2 \text{ (substitute } u=0.8(x-50)\text{)} \\
 &= \frac{760 \times 500^2 - 2 \times 500^3 / 3}{800^2} + 500 \times \left(\frac{325}{1,000} \right)^2 = 219.48
 \end{aligned}$$

Solution 1.14

The conditional densities are uniform:

$$f_X(x | M = m) = \frac{1}{m} \quad \text{for } 0 \leq x \leq m$$

The marginal density function for M is:

$$f_M(1,000) = 0.75 \quad , \quad f_M(2,000) = 0.25$$

So the marginal density function for a randomly selected loss is:

$$\begin{aligned}
 f_X(x) &= \sum_m f_{X,M}(x, m) = \sum_m f_M(m) f_X(x | m) \\
 &= 0.75 f_X(x | m = 1,000) + 0.25 f_X(x | m = 2,000) \\
 &= 0.75 \times \begin{cases} 0.001 & \text{when } 0 \leq x \leq 1,000 \\ 0 & \text{otherwise} \end{cases} + 0.25 \times \begin{cases} 0.0005 & \text{when } 0 \leq x \leq 2,000 \\ 0 & \text{otherwise} \end{cases} \\
 &= \begin{cases} 0.000875 & \text{when } 0 \leq x \leq 1,000 \\ 0.000125 & \text{when } 1,000 < x \leq 2,000 \end{cases}
 \end{aligned}$$

Solution 1.15

We are given that $X | M = m$ is uniform on the interval $[0, m]$. Also, the distribution of M is:

$$\Pr(M = 1,000) = 0.75 \quad \text{and} \quad \Pr(M = 2,000) = 0.25$$

As a result, we have:

$$E[X^k | M = m] = \int_0^m x^k \cdot \frac{1}{m} dx = \frac{m^k}{k+1}$$

So the conditional mean and conditional variance are:

$$E[X | M] = \frac{M}{2}, \quad \text{var}(X | M) = \frac{M^2}{3} - \left(\frac{M}{2}\right)^2 = \frac{M^2}{12}$$

Now apply the double expectation theorem:

$$(i) \quad E[X] = E[E[X | M]] = E\left[\frac{M}{2}\right] = \frac{1,000 \times 0.75 + 2,000 \times 0.25}{2} = 625$$

$$(ii) \quad \begin{aligned} \text{var}(X) &= E[\text{var}(X | M)] + \text{var}(E[X | M]) \\ &= E\left[\frac{M^2}{12}\right] + \text{var}\left(\frac{M}{2}\right) = \frac{E[M^2]}{12} + \frac{E[M^2] - (E[M])^2}{4} \\ &= \frac{4E[M^2] - 3(E[M])^2}{12} \\ &= \frac{4(1,000^2 \times 0.75 + 2,000^2 \times 0.25) - 3(1,000 \times 0.75 + 2,000 \times 0.25)^2}{12} \\ &= 192,708.33 \end{aligned}$$

Solution 1.16

We have:

$$\Pr(N \geq 2) = 1 - \underbrace{\Pr(N = 0)}_{\substack{0.56356 \\ \text{Ex. 9.15}}} - \underbrace{\Pr(N = 1)}_{\substack{0.30840 \\ \text{below}}} = 0.12804$$

$$\begin{aligned} f_N(1) &= \int f_{N, \Theta}(1, \theta) d\theta = \int f_{\Theta}(\theta) f_N(1 | \Theta = \theta) d\theta \\ &= \int_{0.2}^{1.0} \frac{1}{0.8} e^{-\theta} \theta d\theta = \left(-1.25(e^{-\theta}(1+\theta))\right) \Big|_{0.2}^{1.0} = 0.30840 \end{aligned}$$

Solution 1.17

Using the method of transformations, we have:

$$y = x^{-1} \Rightarrow x = y^{-1} \Rightarrow \frac{dx}{dy} = -\frac{1}{y^2}$$

$$f_X(x) = \frac{e^{-x/50}}{50} \quad \text{for } x > 0$$

$$f_Y(y) = f_X(y^{-1}) \left| \frac{dx}{dy} \right| = \frac{e^{-1/50y}}{50y^2} \quad \text{for } y > 0$$

Solution 1.18

For the spliced distribution, we have:

$$f(x) = \begin{cases} 0.8 f_1(x) & \text{when } 0 < x \leq 1,000 \\ 0.2 f_2(x) & \text{when } x > 1,000 \end{cases}$$

$$= \begin{cases} 0.0008 & \text{when } 0 < x \leq 1,000 \\ \frac{0.4 \times 1,000^2}{x^3} & \text{when } x > 1,000 \end{cases}$$

$$E[X] = \int_0^{1,000} 0.0008x \, dx + \int_{1,000}^{\infty} \frac{0.4 \times 1,000^2 x}{x^3} \, dx$$

$$= 400 + \left(-\frac{400,000}{x} \right) \Big|_{1,000}^{\infty} = 400 + 400 = 800$$

Solution 1.19

Since $Y = X | X > 100$, we have:

$$f_Y(y) = \frac{f_X(y)}{\Pr(X > 100)} \quad \text{for } 100 < y < \infty$$

$$= \frac{e^{-y/500} / 500}{e^{-100/500}} \quad \text{for } 100 < y < \infty$$

$$= \frac{\exp\left(-\left(\frac{y-100}{500}\right)\right)}{500} \quad \text{for } 100 < y < \infty$$

The expected value could be computed in several ways:

- $E[Y] = \int_{100}^{\infty} y \exp\left(-\left(\frac{y-100}{500}\right)\right) dy / 500$
- $Y = X | X > 100 = \underbrace{(Y - 100) | X > 100}_{\substack{\text{exponential with mean} \\ 500 \text{ due to memoryless} \\ \text{property of exponentials}}} + 100$
 $\Rightarrow E[Y] = 500 + 100 = 600$

Solution 1.20

The pdf is:

$$f_Y(y) = \begin{cases} f_X(y) & \text{when } y < 1,000 \\ s_X(1,000) & \text{when } y = 1,000 \text{ (discrete part)} \end{cases}$$

$$= \begin{cases} \frac{e^{-y/500}}{500} & \text{when } y < 1,000 \\ e^{-1,000/500} & \text{when } y = 1,000 \text{ (discrete part)} \end{cases}$$

The expected value could be obtained by evaluating the integral:

$$E[Y] = \int_0^{1,000} y \frac{e^{-y/500}}{500} dy + 1,000 e^{-2}$$

In life contingencies (Exam M) this expected value is the same as:

$$\begin{aligned} \ddot{e}_{0:\overline{1,000}|} &= \int_0^{1,000} s_X(x) dx = \int_0^{1,000} e^{-x/500} dx = \left(-500e^{-x/500}\right)\Big|_0^{1,000} \\ &= 500(1 - e^{-2}) = 432.33 \end{aligned}$$