



Construction of Actuarial Models

Second Edition

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Solutions to practice questions – Chapter 10

Solution 10.1

Hypothetical mean: $E[X|\Theta] = \sum_x x \Pr(X = x|\Theta) =$
 $= 0 \times 2\Theta + 1 \times \Theta + 2 \times (1 - 3\Theta) = 2 - 5\Theta$

Pure premium: $E[X] = E[E[X|\Theta]] = E[2 - 5\Theta] = 2 - 5 \times \frac{1/3}{2}$
 $= 7/6$

Solution 10.2

$E[X|\Theta] = 2 - 5\Theta$ (Solution 10.1)

$\text{var}[E(X|\Theta)] = \text{var}[2 - 5\Theta] = 25 \text{var}(\Theta)$

$$= 25 \left[\int_0^{1/3} \theta^2 f_\Theta(\theta) d\theta - \left\{ \int_0^{1/3} \theta f_\Theta(\theta) d\theta \right\}^2 \right]$$

$$= 25 \left[\int_0^{1/3} 3\theta^2 d\theta - \left\{ \int_0^{1/3} 3\theta d\theta \right\}^2 \right]$$

$$= 25 \left[\left(\frac{1}{3} \right)^3 - \left\{ \frac{3}{2} \left(\frac{1}{3} \right)^2 \right\}^2 \right] = \frac{25}{108}$$

$\text{cov}(X_1, X_2) = \text{var}[E(X|\Theta)] = \frac{25}{108}$

Solution 10.3

This is asking for the Bayesian credibility premium $E[X_2|X_1]$.

Solution 10.4

Step 1. Determine $\Pr(X_1 = 0, X_2 = x_2)$

$$\begin{aligned}\Pr(X_1 = 0, X_2 = 0) &= \int_0^{1/3} f_{X_1, X_2, \Theta}(0, 0, \theta) d\theta = \int_0^{1/3} f_{\Theta}(\theta) f_X(0|\theta) f_X(0|\theta) d\theta \\ &= \int_0^{1/3} 3 \times (2\theta) \times (2\theta) d\theta = \frac{4}{27}\end{aligned}$$

$$\begin{aligned}\Pr(X_1 = 0, X_2 = 1) &= \int_0^{1/3} f_{X_1, X_2, \Theta}(0, 1, \theta) d\theta = \int_0^{1/3} f_{\Theta}(\theta) f_X(0|\theta) f_X(1|\theta) d\theta \\ &= \int_0^{1/3} 3 \times (2\theta) \times (\theta) d\theta = \frac{2}{27}\end{aligned}$$

$$\begin{aligned}\Pr(X_1 = 0, X_2 = 2) &= \int_0^{1/3} f_{X_1, X_2, \Theta}(0, 0, \theta) d\theta = \int_0^{1/3} f_{\Theta}(\theta) f_X(0|\theta) f_X(2|\theta) d\theta \\ &= \int_0^{1/3} 3 \times (2\theta) \times (1 - 3\theta) d\theta = \frac{3}{27}\end{aligned}$$

Step 2. Determine $\Pr(X_1 = 0)$

$$\Pr(X_1 = 0) = \sum_{x_2} \Pr(X_1 = 0, X_2 = x_2) = \frac{4 + 2 + 3}{27} = \frac{1}{3}$$

Step 3. Determine $\Pr(X_2 = x_2 | X_1 = 0) = \Pr(X_1 = 0, X_2 = x_2) / \Pr(X_1 = 0)$

$$\Pr(X_2 = 0 | X_1 = 0) = \frac{4/27}{1/3} = \frac{4}{9}$$

$$\Pr(X_2 = 1 | X_1 = 0) = \frac{2/27}{1/3} = \frac{2}{9}$$

$$\Pr(X_2 = 2 | X_1 = 0) = \frac{3/27}{1/3} = \frac{3}{9}$$

Solution 10.5

The Bayesian credibility prediction $E[X_2 | X_1 = 0]$ is the mean of the predictive distribution in Solution 10.4:

$$\begin{aligned}E[X_2 | X_1 = 0] &= \sum_{x_2} x_2 \times \Pr(X_2 = x_2 | X_1 = 0) \\ &= 0 \times \frac{4}{9} + 1 \times \frac{2}{9} + 2 \times \frac{3}{9} = \frac{8}{9}\end{aligned}$$

Solution 10.6

First, apply the double expectation theorem:

$$\begin{aligned}E[X_2 | X_1 = 0] &= E[E[X_2 | \Theta] | X_1 = 0] = E[2 - 5\Theta | X_1 = 0] \quad (\text{Solution 10.1}) \\ &= 2 - 5E[\Theta | X_1 = 0]\end{aligned}$$

Now determine the posterior distribution and the posterior mean:

$$f_{\Theta}(\theta | X_1 = 0) = c f_{\Theta}(\theta) \Pr(X_1 = 0 | \theta) = c \times 3 \times 2\theta$$

$$1 = \int_0^{1/3} f_{\Theta}(\theta | X_1 = 0) d\theta = \frac{3c}{9} \Rightarrow c = 3$$

$$\Rightarrow f_{\Theta}(\theta | X_1 = 0) = 18\theta \text{ for } 0 \leq \theta \leq 1/3$$

$$\Rightarrow E[\Theta | X_1 = 0] = \int_0^{1/3} \theta \times 18\theta d\theta = \frac{6}{27}$$

So the credibility estimate is:

$$\begin{aligned} E[X_2 | X_1 = 0] &= 2 - 5E[\Theta | X_1 = 0] \\ &= 2 - 5 \times \frac{6}{27} = \frac{24}{27} = \frac{8}{9} \end{aligned}$$

Solution 10.7

This is the Poisson/gamma model since an exponential with mean 0.25 is a gamma with $\alpha = 1, \theta = 0.25$:

$$x_1 = x_2 = 0 \Rightarrow$$

$$\alpha^* = \alpha + \sum x_i = 1 + 0 = 1$$

$$\theta^* = \frac{\theta}{1+n\theta} = \frac{0.25}{1 + 2 \times 0.25} = \frac{1}{6}$$

$$E[X_3 | X_1 = 0, X_2 = 0] = \alpha^* \times \theta^* = \frac{1}{6}$$

Solution 10.8

This is the binomial/beta model:

$$\begin{aligned} f_Q(q) &= 4q^3 \text{ for } 0 \leq q \leq 1 \\ &= \frac{\Gamma(5)}{\Gamma(4)\Gamma(1)} q^{4-1} (1-q)^{1-1} \text{ (beta with } a=4, b=1) \end{aligned}$$

Here we have $m = 4$ trials and $X_1 = 2, X_2 = 4$. So the posterior parameters are:

$$a^* = a + \sum x_i = 4 + 6 = 10$$

$$b^* = b + nm - \sum x_i = 1 + 2 \times 4 - 6 = 3$$

The Bayesian credibility estimate of the number of claims in Year 3 is thus:

$$E[X_3 | X_1 = 2, X_2 = 4] = \frac{ma^*}{a^* + b^*} = \frac{4 \times 10}{13} = \frac{40}{13}$$

Solution 10.9

Using the exponential/inverse gamma summary, we have:

$$\begin{aligned}x_1 &= 125, x_2 = 500, x_3 = 75, x_4 = 100, \theta = 200, \alpha = 2 \\ \theta^* &= \theta + \sum x_i = 200 + 800 = 1,000 \\ \alpha^* &= \alpha + n = 2 + 4 = 6 \\ \Rightarrow E[X_4 | X_1 = 125, X_2 = 500, X_3 = 75, X_4 = 100] \\ &= \frac{\theta^*}{\alpha^* - 1} = \frac{1,000}{5} = 200\end{aligned}$$

Solution 10.10

Using the normal/normal summary, we have:

$$\begin{aligned}x_1 &= 125, x_2 = 100, x_3 = 325, x_4 = 50 \\ \mu &= 100, \sigma_1^2 = 100, \sigma_2^2 = 225 \\ \mu^* &= \left(\frac{\sum x_i}{\sigma_1^2} + \frac{\mu}{\sigma_2^2} \right) / \left(\frac{n}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) \\ &= \left(\frac{600}{100} + \frac{100}{225} \right) / \left(\frac{4}{100} + \frac{1}{225} \right) = 145 \\ E[X_4 | X_1 = 125, X_2 = 100, X_3 = 325, X_4 = 50] &= \mu^* = 145\end{aligned}$$

Solution 10.11

$$\begin{aligned}\Pr(\text{Type A} | S_1 = 300) &= c \underbrace{\Pr(\text{Type A})}_{0.5} \underbrace{\Pr(S_1 = 300 | \text{Type A})}_{\substack{3 \text{ claims of } 100 \text{ or } 2 \text{ claims} \\ \text{equal to } 100, 200 \text{ or } 200, 100}} \\ &= c \times 0.5 \times \left(e^{-0.6} \times \frac{0.6^3}{3!} \times 0.5^3 + 2 \times e^{-0.6} \times \frac{0.6^2}{2!} \times 0.5^2 \right) \\ &= 0.02593c\end{aligned}$$

$$\begin{aligned}\Pr(\text{V B} | S_1 = 300) &= c \underbrace{\Pr(\text{Type B})}_{0.5} \underbrace{\Pr(S_1 = 300 | \text{Type B})}_{1 \text{ claim equal to } 300} \\ &= c \times 0.5 \times \left(e^{-0.4} \times \frac{0.4^1}{1!} \times 0.5 \right) \\ &= 0.06703c\end{aligned}$$

$$1 = 0.02593c + 0.06703c \Rightarrow c = 10.75693$$

$$\Rightarrow \Pr(\text{Type A} | S_1 = 300) = 0.02593c = 0.27894$$

Solution 10.12

The Bayesian credibility estimate for the total annual claim in Year 2 is:

$$\begin{aligned} E[S_2 | S_1 = 300] &= E[E[S_2 | \text{Class}] | S_1 = 300] \\ &= E[S_2 | \text{Class A}] \times \Pr(\text{Class A} | S_1 = 300) \\ &\quad + E[S_2 | \text{Class B}] \times \Pr(\text{Class B} | S_1 = 300) \end{aligned}$$

So we need to determine the expected annual claim for each class and then take a weighted average using the posterior probabilities from Solution 10.11:

$$\begin{aligned} E[S | \text{Class A}] &= E[N | \text{Class A}] E[X | \text{Class A}] \\ &= 0.6 \times 150 = 90 \\ E[S | \text{Class B}] &= E[N | \text{Class B}] E[X | \text{Class B}] \\ &= 0.4 \times 250 = 100 \end{aligned}$$

Using the results of Solution 10.11, the credibility premium is:

$$\begin{aligned} E[S_2 | S_1 = 200] &= \underbrace{E[S_2 | \text{Class A}]}_{90} \times \underbrace{\Pr(\text{Class A} | S_1 = 300)}_{0.27894} \\ &\quad + \underbrace{E[S_2 | \text{Class B}]}_{100} \times \underbrace{\Pr(\text{Class B} | S_1 = 300)}_{0.72106} = 97.21 \end{aligned}$$

Solution 10.13

$$\begin{aligned} \Pr(\Theta = 200 | X_1 = 350) &= c \Pr(\Theta = 200) f_X(350 | \Theta = 200) \\ &= c \times 0.8 \times \frac{200}{(200 + 350)^2} = 0.000529c \\ \Pr(\Theta = 300 | X_1 = 350) &= c \Pr(\Theta = 300) f_X(350 | \Theta = 300) \\ &= c \times 0.2 \times \frac{300}{(300 + 350)^2} = 0.000142c \end{aligned}$$

So the probability that $\theta = 200$ for this policy is determined as follows:

$$\begin{aligned} 1 &= 0.000529c + 0.000142c \Rightarrow c = 1,490.45190 \\ \Rightarrow \Pr(\Theta = 200 | X_1 = 350) &= 0.000529c = 0.78834 \end{aligned}$$

Solution 10.14

For the given conditional PDF, the conditional survival function is:

$$f_X(x | \Theta = \theta) = \frac{\theta}{(\theta+x)^2} \text{ for } x > 0$$

$$\Rightarrow s_X(x | \Theta = \theta) = \int_x^\infty \frac{\theta}{(\theta+t)^2} dt = \frac{\theta}{\theta+x}$$

We are asked to calculate $\Pr(X_2 > 300 | X_1 = 350)$:

$$\begin{aligned} \Pr(X_2 > 300 | X_1 = 350) &= \sum_{\theta} \Pr(X_2 > 300 | \Theta = \theta) \Pr(\Theta = \theta | X_1 = 350) \\ &= \sum_{\theta} s_X(300 | \Theta = \theta) \Pr(\Theta = \theta | X_1 = 350) \\ &= \underbrace{s_X(300 | \Theta = 200)}_{\frac{200}{200+300}} \underbrace{\Pr(\Theta = 200 | X_1 = 350)}_{\text{Solution 10.13: } 0.78834} \\ &\quad + \underbrace{s_X(300 | \Theta = 300)}_{\frac{300}{300+300}} \underbrace{\Pr(\Theta = 300 | X_1 = 350)}_{\text{Solution 10.13: } 0.21166} \\ &= 0.42117 \end{aligned}$$

Solution 10.15

$$\begin{aligned} \Pr(\Theta = 0.15 | X_1 = \dots = X_{10} = 0) &= c \Pr(\Theta = 0.15) \Pr(X = 0 | \theta = 0.15)^{10} \\ &= c \times 0.2 \times e^{-0.15 \times 10} = 0.04463c \end{aligned}$$

$$\begin{aligned} \Pr(\Theta = 0.30 | X_1 = \dots = X_{10} = 0) &= c \Pr(\Theta = 0.30) \Pr(X = 0 | \theta = 0.30)^{10} \\ &= c \times 0.8 \times e^{-0.30 \times 10} = 0.03983c \end{aligned}$$

Since the probabilities must sum to one we have:

$$\Pr(\Theta = 0.15 | X_1 = \dots = X_{10} = 0) = 0.04463c = 0.52840$$

Solution 10.16

$$\begin{aligned} \Pr(\Theta = 0.15 | X_1 = \dots = X_n = 0) &= c \Pr(\Theta = 0.15) \Pr(X = 0 | \theta = 0.15)^n \\ &= c \times 0.2 \times e^{-0.15 \times n} \end{aligned}$$

$$\begin{aligned} \Pr(\Theta = 0.30 | X_1 = \dots = X_n = 0) &= c \Pr(\Theta = 0.30) \Pr(X = 0 | \theta = 0.30)^n \\ &= c \times 0.8 \times e^{-0.30 \times n} \end{aligned}$$

$$\begin{aligned}\Pr(\Theta = 0.15 \mid X_1 = \dots = X_n = 0) &= \frac{c \times 0.2 \times e^{-0.15 \times n}}{c \times 0.2 \times e^{-0.15 \times n} + c \times 0.8 \times e^{-0.30 \times n}} \\ &= \frac{0.2}{0.2 + 0.8 e^{-0.15 \times n}} \xrightarrow{n \rightarrow \infty} \frac{0.2}{0.2 + 0.8 \times 0} = 1\end{aligned}$$

Solution 10.17

Using the normal/normal summary, we have:

$$\begin{aligned}x_1 &= 125, x_2 = 100, x_3 = 325, x_4 = 50 \\ \mu &= 100, \sigma_1^2 = 100, \sigma_2^2 = 225 \\ \mu^* &= \left(\frac{\sum x_i}{\sigma_1^2} + \frac{\mu}{\sigma_2^2} \right) \bigg/ \left(\frac{n}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) \\ &= \left(\frac{600}{100} + \frac{100}{225} \right) \bigg/ \left(\frac{4}{100} + \frac{1}{225} \right) = 145 \\ \sigma^{*2} &= 1 \bigg/ \left(\frac{n}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) = 1 \bigg/ \left(\frac{4}{100} + \frac{1}{225} \right) = 22.50\end{aligned}$$

We are asked to determine the posterior probability:

$$\begin{aligned}\Pr(\Theta > 150 \mid x_1 = 125, x_2 = 100, x_3 = 325, x_4 = 50) \\ &= \Pr\left(N(\mu^* = 145, \sigma^{*2} = 22.5) > 150\right) \\ &= 1 - \Phi\left(\frac{150 - 145}{\sqrt{22.5}}\right) = 1 - \Phi(1.054) = 1 - 0.8541 \\ &= 0.1459\end{aligned}$$

Solution 10.18

This is the Poisson/gamma model since an exponential with mean 0.25 is a gamma with $\alpha = 1, \theta = 0.25$:

$$\begin{aligned}x_1 = x_2 = 0 &\Rightarrow \\ \alpha^* &= \alpha + \sum x_i = 1 + 0 = 1 \\ \theta^* &= \frac{\theta}{1 + n\theta} = \frac{0.25}{1 + 2 \times 0.25} = \frac{1}{6}\end{aligned}$$

So the posterior distribution of the Poisson parameter Λ is a gamma distribution with the above parameters. So in fact it is exponential with mean 1/6. So the posterior probability that $\Lambda > 0.25$ is:

$$e^{-0.25/(1/6)} = e^{-1.5} = 0.22313$$

Solution 10.19

Step 1. Determine the posterior distribution:

$$\Pr(\Theta = \theta | X_1 = 5) = c \Pr(\Theta = \theta) f_X(5 | \Theta = \theta) = c \Pr(\Theta = \theta) \times \frac{e^{-5/\theta}}{\theta} \Rightarrow$$

$$\Pr(\Theta = 2 | X_1 = 5) = c \times 0.5 \times \frac{e^{-5/2}}{2} = 0.02052c$$

$$\Pr(\Theta = 4 | X_1 = 5) = c \times 0.5 \times \frac{e^{-5/4}}{4} = 0.03581c$$

$$1 = 0.02052c + 0.03581c \Rightarrow c = 17.75116 \Rightarrow$$

$$\Pr(\Theta = 2 | X_1 = 5) = 0.36428$$

$$\Pr(\Theta = 4 | X_1 = 5) = 0.63572$$

Step 2. Determine $\Pr(X_2 \geq 10 | X_1 = 5)$:

$$\Pr(X_2 \geq 10 | X_1 = 5) = \sum_{\theta} \Pr(X_2 > 10 | \theta) \Pr(\Theta = \theta | X_1 = 5)$$

$$= \sum_{\theta} e^{-10/\theta} \Pr(\Theta = \theta | X_1 = 5)$$

$$= e^{-10/2} \Pr(\Theta = 2 | X_1 = 5) + e^{-10/4} \Pr(\Theta = 4 | X_1 = 5)$$

$$= 0.05464$$

Solution 10.20

It helps to recognize that this is the exponential/inverse gamma combination:

$$\text{exponential PDF: } f_X(x | \Lambda = \lambda) = \frac{e^{-x/\lambda}}{\lambda} \text{ for } x > 0$$

$$\text{inverse gamma PDF: } f_{\Lambda}(\lambda) = \frac{\theta^{\alpha} e^{-\theta/\lambda}}{\lambda^{\alpha+1} \Gamma(\alpha)} = c \times \frac{e^{-\theta/\lambda}}{\lambda^{\alpha+1}} \text{ for } \lambda > 0$$

Comparing the inverse gamma PDF with the given formula, we see that the prior is inverse gamma with parameters given as follows:

$$f_{\Lambda}(\lambda) = e^{-1/\lambda} / \lambda^2 = c \times \frac{e^{-\theta/\lambda}}{\lambda^{\alpha+1}} \Rightarrow \theta = \alpha = 1$$

We are given $n=1$ observation with $x_1=5$. So according to the exponential/inverse gamma summary, the posterior inverse gamma parameters are:

$$\theta^* = \theta + \sum x_i = 1 + 5 = 6$$

$$\alpha^* = \alpha + n = 1 + 1 = 2$$

$$\Rightarrow f_{\Lambda}(\lambda | X_1 = 5) = \frac{6^2 e^{-6/\lambda}}{\lambda^3} \text{ for } \lambda > 0$$

The conditional exponential survival function is $\Pr(X > x | \Lambda = \lambda) = e^{-x/\lambda}$, so we have:

$$\begin{aligned}
 \Pr(X_2 \geq 10 | X_1 = 5) &= \int_0^{\infty} \Pr(X_2 \geq 10 | \Lambda = \lambda) f_{\Lambda}(\lambda | X_1 = 5) d\lambda \\
 &= \int_0^{\infty} e^{-10/\lambda} \times \frac{6^2 e^{-6/\lambda}}{\lambda^3} d\lambda \\
 &= \int_0^{\infty} \frac{6^2 e^{-16/\lambda}}{\lambda^3} d\lambda \\
 &= \frac{6^2}{16^2} \times \underbrace{\int_0^{\infty} \frac{16^2 e^{-16/\lambda}}{\lambda^3} d\lambda}_{\substack{\text{inverse gamma PDF} \\ \theta=16, \alpha=2}} = \frac{6^2}{16^2} \times 1 = 0.140625
 \end{aligned}$$