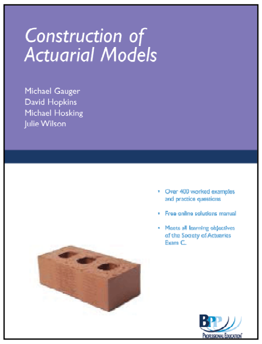

Construction of Actuarial Models

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Solutions to practice questions – Chapter 11

Solution 11.1

We first find the expected claims outgo for risks of each type. For Type I we have:

$$E[X \mid \theta = 1] = 1 \times 0.8 + 4 \times 0.2 = 1.6$$

For risks of Type II we have:

$$E[X \mid \theta = 2] = 1 \times 0.3 + 4 \times 0.7 = 3.1$$

The collective premium is the overall mean, taking account of the likelihood that a risk might be of a particular type. Since 60% of the portfolio are Type I risks, we have:

$$\mu = 0.6 \times 1.6 + 0.4 \times 3.1 = 2.2$$

So the collective premium is 2.2 for this portfolio. This is the amount we would charge for a risk if we had no prior information about it.

Solution 11.2

We now need the variance of the claim amount for each type of risk:

$$\text{var}(X \mid \theta = 1) = 1^2 \times 0.8 + 4^2 \times 0.2 - 1.6^2 = 1.44$$

for the Type I risks, and:

$$\text{var}(X \mid \theta = 2) = 1^2 \times 0.3 + 4^2 \times 0.7 - 3.1^2 = 1.89$$

for the Type II risks.

We can now calculate the expected value of the process variance:

$$\upsilon = E[\upsilon(\theta)] = 0.6 \times 1.44 + 0.4 \times 1.89 = 1.62$$

The variance of the hypothetical means is:

$$a = \text{var}[\mu(\theta)] = 1.6^2 \times 0.6 + 3.1^2 \times 0.4 - 2.2^2 = 0.54$$

Solution 11.3

The credibility factor will be:

$$Z = \frac{n}{n+\upsilon/a} = \frac{3}{3+\frac{1.62}{0.54}} = 0.5$$

For this particular policyholder, the sample mean is $\bar{x} = 3$. So the Buhlmann premium is:

$$CP = 0.5\times 3 + 0.5\times 2.2 = 2.6$$

Solution 11.4

For this other risk, the sample mean is 1. So the credibility premium is now:

$$CP = 0.5\times 1 + 0.5\times 2.2 = 1.6$$

Solution 11.5

If we adopt a Bayesian approach here, we proceed as follows.

We use Bayes' Theorem to turn the probabilities around. Looking first at the policyholder in Question 3, who has a total claim amount of 9 over three years, we will use S for the aggregate claim amount over the three year period. The posterior probabilities for θ can be calculated using:

$$\Pr(\theta=1 \mid S=9)=\frac{\Pr(S=9 \mid \theta=1)\Pr(\theta=1)}{\Pr(S=9 \mid \theta=1)\Pr(\theta=1)+\Pr(S=9 \mid \theta=2)\Pr(\theta=2)}$$

If $\theta=1$, we can get an aggregate claim amount of 9 in any of the three ways:

1, 4, 4 or 4, 1, 4 or 4, 4, 1

The probability of obtaining any of these claims patterns if $\theta=1$ is $0.8\times0.2\times0.2=0.032$. So the total probability is:

$$\Pr(S=9 \mid \theta=1)=3\times0.032=0.096$$

Similarly, if $\theta=2$, the corresponding probability is:

$$\Pr(S=9 \mid \theta=2)=3\times0.3\times0.7^2=0.441$$

We now use Bayes' Theorem:

$$\begin{aligned}\Pr(\theta=1 \mid S=9)&=\frac{\Pr(S=9 \mid \theta=1)\Pr(\theta=1)}{\Pr(S=9 \mid \theta=1)\Pr(\theta=1)+\Pr(S=9 \mid \theta=2)\Pr(\theta=2)}\\&=\frac{0.096\times0.6}{0.096\times0.6+0.441\times0.4}=0.24615\end{aligned}$$

Similarly, by subtraction:

$$\Pr(\theta=2 \mid S=9)=1-0.24615=0.75385$$

So the expected claim amount for this policy for the coming year is (using the expected claim amounts we calculated earlier):

$$1.6\times0.24615+3.1\times0.75385=2.7308$$

We can calculate the probabilities for the second policyholder similarly. If his aggregate claim amount is 3 over the three year period, the claims amounts must be 1, 1, 1. The probability of getting this sequence of amounts is $0.8^3=0.512$ if $\theta=1$, and $0.3^3=0.027$ if $\theta=2$. So the corresponding posterior probabilities for θ for the second policyholder are:

$$\begin{aligned}\Pr(\theta=1 \mid S=3)&=\frac{\Pr(S=3 \mid \theta=1)\Pr(\theta=1)}{\Pr(S=3 \mid \theta=1)\Pr(\theta=1)+\Pr(S=3 \mid \theta=2)\Pr(\theta=2)}\\&=\frac{0.512\times0.6}{0.512\times0.6+0.027\times0.4}=0.96604\end{aligned}$$

and the posterior probability that $\theta=2$ is $1-0.96604=0.03396$. So the expected outgo for the second policyholder is:

$$0.96604\times1.6+0.03396\times3.1=1.65094$$

or about 1.65.

Solution 11.6

Let X be the number of claims made in a year by this policy. We know that $X|q$ has a $Binomial(m,q)$ distribution. So using the formulas for the mean and variance of a binomial distribution:

$$E\left[X|q\right]=mq \qquad \text{and:} \qquad \text{var}(X|q)=mq(1-q)$$

We also need the corresponding formulas for the beta distribution. The mean of a beta distribution is:

$$\mu=\frac{a}{a+b}$$

The variance of a beta distribution is:

$$\sigma^2=\frac{ab}{(a+b)^2(a+b+1)}$$

We will also need the second non-central moment of the beta distribution, as we will see in a minute. So, for the beta distribution:

$$E\left[Y^2\right]=\mu^2+\sigma^2=\left(\frac{a}{a+b}\right)^2+\frac{ab}{(a+b)^2(a+b+1)}=\frac{a(a+1)}{(a+b)(a+b+1)}$$

We can now calculate our Buhlmann parameters.

The collective premium is:

$$\mu=E[E\left[X|q\right]]=E\left[mq\right]=mE\left[q\right]=\frac{ma}{a+b}$$

The expected value of the process variance is:

$$\upsilon=E[\text{var}(X|q)]=E[mq(1-q)]=m\left(E\left[q\right]-E\left[q^2\right]\right)$$

From the results above for the beta distribution, we know that $E\left[q\right]=\dfrac{a}{a+b}$ and $E\left[q^2\right]=\dfrac{a(a+1)}{(a+b)(a+b+1)}$. So:

$$\upsilon=m\left(E\left[q\right]-E\left[q^2\right]\right)=m\left(\frac{a}{a+b}-\frac{a(a+1)}{(a+b)(a+b+1)}\right)=m\left(\frac{ab}{(a+b)(a+b+1)}\right)$$

Similarly, the variance of the hypothetical means is:

$$a=\text{var}\left(E\left[X|q\right]\right)=\text{var}\left(mq\right)=m^2\,\text{var}(q)=m^2\frac{ab}{(a+b)^2(a+b+1)}$$

We can now calculate the credibility factor, using:

$$Z=\frac{n}{n+\upsilon/a}=\frac{n}{n+\dfrac{a+b}{m}}=\frac{mn}{mn+a+b}$$

The formula for the credibility premium is:

$$CP=Z\,\overline{x}+(1-Z)\mu=\frac{mn}{mn+a+b}\overline{x}+\frac{a+b}{mn+a+b}\times\frac{ma}{a+b}=\frac{m(a+n\overline{x})}{mn+a+b}$$

Solution 11.7

We now have, using the notation in the previous question:

$$m = 2 \qquad n = 10 \qquad \sum x_i = 6 \qquad a = 4 \qquad b = 6$$

So the credibility premium is:

$$CP = \frac{2(4+6)}{20+4+6} = \frac{2}{3}$$

Solution 11.8

This does look like a weighted average. The prior expected number of claims per year is $2 \times 0.4 = 0.8$, and the actual sample mean number of claims per year is 0.6. Our credibility premium lies in between these two values, and is closer to the sample mean, suggesting that we are using a credibility factor that exceeds 0.5. In fact we can calculate the credibility factor (using the formula in the previous solution), to be 2/3.

Solution 11.9

We first need the parameters of the gamma distirbution. Using the formulas for the mean and variance of the gamma distribution, we have:

$$\alpha\beta = 0.2 \qquad \alpha\beta^2 = 0.4$$

Solving these equations, we find that $\alpha = 0.1$, $\beta = 2$.

Using the results given in the chapter for the Poisson/gamma model, we have:

$$\mu = E[\mu(\lambda)] = \alpha\beta = 0.2$$

$$\upsilon = E[\upsilon(\lambda)] = \alpha\beta = 0.2$$

$$a = \text{var}\big(\mu(\lambda)\big) = \alpha\beta^2 = 0.4$$

So the credibility factor is:

$$Z = \frac{n\beta}{n\beta + 1} = \frac{20}{21}$$

The sample mean is 0.7, so the credibility premium is:

$$CP = \frac{20}{21} \times 0.7 + \frac{1}{21} \times 0.2 = 0.6762$$

Solution 11.10

Using notation consistent with that used in the previous chapter, we assume that claim amounts $X \mid \theta$ have a normal distribution $N(\theta, \sigma_1^2)$. Considered as a random variable, θ has another normal distribution $N(\mu, \sigma_2^2)$. So the collective premium is:

$$E[X] = E[E[X \mid \theta]] = E[\theta] = \mu$$

Similarly we have:

$$\upsilon = E[\text{var}(X \mid \theta)] = E\left[\sigma_1^2\right] = \sigma_1^2$$

and:

$$a = \text{var}\left(E[X \mid \theta]\right) = \text{var}(\theta) = \sigma_2^2$$

So the credibility factor is:

$$Z = \frac{n}{n + \upsilon / a} = \frac{n}{n + \sigma_1^2 / \sigma_2^2}$$

This is the same formula as the one we got for the normal/normal model when we were using the Bayesian approach.

So the credibility premium is also the same:

$$CP = \frac{n\bar{x}\sigma_2^2 + \mu\sigma_1^2}{\sigma_1^2 + n\sigma_2^2}$$

Solution 11.11

The number of claims made by a policyholder, $N \mid \theta$, has a $Poisson(\theta)$ distribution. Treated as a random variable, the distribution for θ is:

$$\Pr(\theta = 0.1) = \Pr(\theta = 0.2) = ½$$

So we can now calculate the conditional means and variances:

$$\mu = E[E[N \mid \theta]] = E[\theta] = 0.15$$

$$\upsilon = E[\text{var}(N \mid \theta)] = E[\theta] = 0.15$$

$$a = \text{var}\left(E[N \mid \theta]\right) = \text{var}(\theta) = 0.1^2 \times ½ + 0.2^2 \times ½ - 0.15^2 = 0.0025$$

So the credibility factor is now:

$$Z = \frac{3}{3 + \dfrac{0.15}{0.0025}} = \frac{3}{63}$$

and the credibility premium is:

$$\frac{3}{63} \times 0 + \frac{60}{63} \times 0.15 = 0.1429$$

Solution 11.12

This is again the beta/binomial model, with a beta prior distribution with parameters $a=3$ and $b=2$. So using the results from Solution 11.6 with $a=3$, $b=2$, $m=3$, $n=10$ and $\sum x_i = 4$, we have:

$$CP = \frac{m(a+n\bar{x})}{mn+a+b} = \frac{21}{35} = 0.6$$

Solution 11.13

From Solution 11.9, we know that $\alpha = 0.1$ and $\beta = 2$. Also from Solution 11.9:

$$\mu = \alpha\beta = 0.2$$

$$\upsilon = \alpha\beta = 0.2$$

$$a = \alpha\beta^2 = 0.4$$

The credibility factor is now:

$$Z = \frac{185}{185 + \frac{1}{2}} = 0.9973$$

The mean number of claims per year from the portfolio is:

$$\frac{15}{185} = 0.08108$$

So the credibility premium is:

$$0.9973 \times 0.08108 + 0.0027 \times 0.2 = 0.0814$$

Solution 11.14

Let us first find the mean and variance of θ:

$$E[\theta]=\int_0^1 3\theta^3 d\theta=\left[3\theta^4/4\right]_0^1=\frac{3}{4}$$

$$E\left[\theta^2\right]=\int_0^1 3\theta^4 d\theta=\left[3\theta^5/5\right]_0^1=\frac{3}{5}$$

$$\text{var}(\theta)=\frac{3}{5}-\left(\frac{3}{4}\right)^2=\frac{3}{80}$$

We can now find the Buhlmann-Straub parameters:

$$\mu=E[N]=E[E[N\mid\theta]]=E[5\theta]=5\times3/4=\frac{15}{4}$$

$$\upsilon=E[\text{var}(N\mid\theta)]=E[5\theta(1-\theta)]=5\left(E[\theta]-E\left[\theta^2\right]\right)=5\left(\frac{3}{4}-\frac{3}{5}\right)=\frac{3}{4}$$

$$a=\text{var}\left(E[N\mid\theta]\right)=\text{var}(5\theta)=25\times\frac{3}{80}=\frac{15}{16}$$

So the credibility factor is:

$$Z=\frac{45}{45+\frac{4}{5}}=0.98253$$

and the credibility premium is:

$$CP=0.98253\times\frac{41}{45}+0.01747\times\frac{15}{4}=0.9607$$

Solution 11.15

For Type I policies we have:

$$E[N\mid\text{I}]=0.8+0.2=1$$

$$\text{var}(N\mid\text{I})=0.8+0.4-1^2=0.2$$

For Type II policies we have:

$$E[N\mid\text{II}]=0.1+0.2=0.3$$

$$\text{var}(N\mid\text{II})=0.1+0.4-0.3^2=0.41$$

For Type III policies we have:

$$E[N\mid\text{III}]=0.1+1.6=1.7$$

$$\text{var}(N\mid\text{III})=0.1+3.2-1.7^2=0.41$$

It will be useful for the remaining solutions to arrange these conditional values in a table alongside the distribution of the risk parameter (type):

Type	Probability	$E[N \mid \text{type}]$	$\text{var}(N \mid \text{type})$
I	0.50	1.0	0.20
II	0.25	0.3	0.41
III	0.25	1.7	0.41

Solution 11.16

In the table above we have given the distribution of the risk type as well as the conditional means and variances.

The expected value of the hypothetical means is:

$$\mu = E\left[E[N \mid \text{type}]\right] = 0.5 \times 1 + 0.25 \times 0.3 + 0.25 \times 1.7 = 1$$

The expected value of the process variance is:

$$\upsilon = E\left[\text{var}(N \mid \text{type})\right] = 0.5 \times 0.2 + 0.25 \times 0.41 + 0.25 \times 0.41 = 0.305$$

The variance of the hypothetical means is:

$$a = \text{var}\left(E[N \mid \text{type}]\right) = 0.5 \times 1^2 + 0.25 \times 0.3^2 + 0.25 \times 1.7^2 - 1^2 = 0.245$$

Solution 11.17

We now need the average claims experience for this class. The mean number of claims per policy year in the sample is:

$$\frac{40+25+10}{50+60+80} = \frac{75}{190} = 0.39474$$

The credibility factor is:

$$Z = \frac{190}{190 + \frac{0.305}{0.245}} = 0.99349$$

So the expected number of claims is:

$$50(0.99349 \times 0.39474 + 0.00651 \times 1) = 19.934$$

Solution 11.18

We now have a distribution for type that is given by:

$$\Pr(\text{type} = \text{I}) = \Pr(\text{type} = \text{II}) = \Pr(\text{type} = \text{III}) = \frac{1}{3}$$

Let's insert here the modified table that we constructed above:

Type	Probability	$E\left[N \mid \text{type}\right]$	$\text{var}\left(N \mid \text{type}\right)$
I	1/3	1.0	0.20
II	1/3	0.3	0.41
III	1/3	1.7	0.41

So the expected value of the hypothetical means is:

$$\mu = \frac{1}{3} \times 1 + \frac{1}{3} \times 0.3 + \frac{1}{3} \times 1.7 = 1$$

The expected value of the process variance is:

$$\upsilon = \frac{1}{3} \times 0.2 + \frac{1}{3} \times 0.41 + \frac{1}{3} \times 0.41 = 0.34$$

The variance of the hypothetical means:

$$a = \frac{1}{3} \times 1^2 + \frac{1}{3} \times 0.3^2 + \frac{1}{3} \times 1.7^2 - 1^2 = 0.3267$$

So we now get:

$$Z = \frac{190}{190 + \dfrac{0.34}{0.3267}} = 0.99455$$

and the credibility premium is:

$$50(0.99455 \times 0.39474 + 0.00545 \times 1) = 19.902$$

Solution 11.19

The two credibility premiums are very similar. This is because:

(1) the credibility factor is very high in both cases. We are primarily using the data from the actual risk, and the contribution from the expected value of the hypothetical means is very small. The data from the actual risk is unchanged in the two cases.

(2) because of the symmetry of the situation, the expected value of the hypothetical means is 1 in both cases. So it is only the credibility factors that are changing, and they do not change by much.

Solution 11.20

For a policy of Type IV we have:

$$E[N] = \frac{1}{3}(0+1+2) = 1$$

$$\text{var}(N) = \frac{1}{3}(0^2+1^2+2^2) - 1^2 = \frac{2}{3}$$

We can now see how this example will be different from that in Question 11.18.

The expected value of the hypothetical mean will be the same in both cases.

Since the new variance of $\frac{2}{3}$ is much bigger than the other process variances, the expected value of the process variance will increase.

Since the new mean of 1 is exactly equal to the overall mean, the variance of the hypothetical means will decrease, since there is no extra variability, and the overall variability is spread over four risks instead of three. So, if VHM has decreased and EPV has increased, the ratio υ / a will increase, and the value of Z will decrease.

So the credibility factor will be (a little) smaller than it was in Solution 11.18.