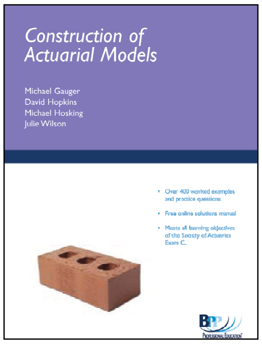

Construction of Actuarial Models

By J. Wilson, M. Hosking, D. Hopkins, and M. Gauger

Published by BPP Professional Education

Solutions to practice questions – Chapter 12

Solution 12.1

Using the parameter values already found in the chapter, the estimate of the credibility premium is:

$$\hat{Z}\,\bar{X}_3 + (1-\hat{Z})\hat{\mu} = 0.5136 \times 55.75 + 0.4864 \times 57.9167 = 56.8$$

Solution 12.2

From this data we have:

$$r = 5 \qquad n = 4$$

We can calculate $\bar{X}_i$ and $\bar{X}$:

$$\bar{X}_1 = \frac{18}{4} = 4.5 \qquad \bar{X}_2 = \frac{25}{4} = 6.25$$

$$\bar{X}_3 = \frac{21}{4} = 5.25 \qquad \bar{X}_4 = \frac{23}{4} = 5.75 \qquad \bar{X}_5 = \frac{18}{4} = 4.5$$

$$\bar{X} = \frac{1}{5}(4.5 + 6.25 + 5.25 + 5.75 + 4.5) = 5.25$$

$$\hat{\mu} = \bar{X} = 5.25$$

The required formula for $\hat{\upsilon}$ is:

$$\hat{\upsilon} = \frac{1}{r(n-1)} \sum_{i=1}^{r} \sum_{j=1}^{n} (X_{ij} - \bar{X}_i)^2$$

$$\hat{\upsilon} = \frac{1}{5(4-1)} \left\{ \begin{array}{l} (3-4.5)^2 + (8-4.5)^2 + (2-4.5)^2 + (5-4.5)^2 \\ +(8-6.25)^2 + (5-6.25)^2 + (10-6.25)^2 + (2-6.25)^2 \\ +(7-5.25)^2 + (0-5.25)^2 + (9-5.25)^2 + (5-5.25)^2 \\ +(2-5.75)^2 + (3-5.75)^2 + (11-5.75)^2 + (7-5.75)^2 \\ +(6-4.5)^2 + (8-4.5)^2 + (4-4.5)^2 + (0-4.5)^2 \end{array} \right\} = 12.55$$

The required formula for $\hat{a}$ is:

$$\hat{a} = \frac{1}{r-1}\sum_{i=1}^{r}(\bar{X}_i - \bar{X})^2 - \frac{\hat{\upsilon}}{n}$$

$$\hat{a} = \frac{1}{4}\left\{(4.5-5.25)^2 + (6.25-5.25)^2 + 0^2 + (5.75-5.25)^2 + (4.5-5.25)^2\right\} - \frac{12.55}{4}$$
$$= -2.54375$$

Since $\hat{a}$ is less than 0, the estimate of the credibility factor is 0.

Solution 12.3

In the previous solution, the credibility factor is zero. Why is this?

First, take a look at the data in the table in the previous question. The variation within the data for each individual policyholder is quite great. Also, there is a significant overlap between the data from the different insurers. This means that it is quite difficult for the model to detect differences in the experience of the different insurers in the group. When this happens, there is a tendency for the model to "give up", and assign the same premium (the overall mean) to all the different insurers. This is what happens when we use a credibility value of zero.

Looked at from a slightly different perspective, the parameter $\upsilon = E\left[\upsilon(\Theta_i)\right]$ is a measure of the variability *within* the data from each individual insurer in the group. $a = \mathrm{var}\left(\mu(\Theta_i)\right)$ is a measure of the variability *between* the different insurers' experience. When υ is much bigger than a, the variability within individual insurers swamps any detectable differences between the experiences. So the model assigns a single credibility premium to all the insurers in the group.

Solution 12.4

The estimate of the credibility factor is:

$$\hat{Z}_2 = \frac{162}{162 + 149.118} = 0.5207$$

The estimate of the credibility premium per individual for group 2 is:

$$\hat{Z}_2\bar{X}_2 + (1-\hat{Z}_2)\hat{\mu} = 0.5207 \times 4,202.160 + (1-0.5207)\times 4,063.511 = 4,136$$

So for the whole group, the premium is:

$$4,136 \times 47 = 194,378$$

Solution 12.5

From the question:

$$r = 2 \qquad\qquad n_1 = n_2 = 4$$

We can then calculate the necessary values of m_i and m:

$$m_1 = 11 + 8 + 15 + 16 = 50$$
$$m_2 = 5 + 9 + 8 + 11 = 33$$
$$m = 50 + 33 = 83$$

We need the X_{ij} values. These are the values in the table. This enables us to find $\bar{X}_i$:

$$\bar{X}_1 = \frac{1}{m_1}\sum_j m_{1j}X_{1j} = \frac{82\times 11 + 48\times 8 + 100\times 15 + 110\times 16}{50} = 90.92$$
$$\bar{X}_2 = \frac{1}{m_2}\sum_j m_{2j}X_{2j} = \frac{22\times 5 + 82\times 9 + 56\times 8 + 103\times 11}{33} = 73.60606$$

The overall mean is then:

$$\begin{aligned}\bar{X} &= \frac{1}{m}\sum_i m_i\bar{X}_i \\ &= \frac{1}{83}(50\times 90.92 + 33\times 73.60606) = 84.0361\end{aligned}$$

This means that $\hat{\mu} = 84.0361$.

The formula for $\hat{\upsilon}$ is:

$$\hat{\upsilon} = \frac{1}{\sum_{i=1}^{r}(n_i - 1)}\sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{ij}(X_{ij} - \bar{X}_i)^2$$

So:

$$\begin{aligned}\hat{\upsilon} &= \frac{1}{3+3}\left\{\begin{matrix} 11(82-90.92)^2 + 8(48-90.92)^2 + 15(100-90.92)^2 \\ +16(110-90.92)^2 + 5(22-73.60606)^2 + 9(82-73.60606)^2 \\ +8(56-73.60606)^2 + 11(103-73.60606)^2 \end{matrix}\right\} \\ &= 8{,}101.2598\end{aligned}$$

The formula for $\hat{a}$ is:

$$\hat{a} = \left(m - \frac{1}{m}\sum_{i=1}^{r} m_i^2\right)^{-1}\left[\sum_{i=1}^{r} m_i(\bar{X}_i - \bar{X})^2 - \hat{\upsilon}(r-1)\right]$$

So:

$$\hat{a} = \frac{50(90.92-84.0361)^2 + 33(73.60606-84.0361)^2 - 8{,}101.2598}{\left(83 - \frac{1}{83}(50^2 + 33^2)\right)} < 0$$

The credibility factor is then taken to be zero, giving the estimate of the premium as $\hat{\mu} = 84$.

Solution 12.6

The table now reads:

		Year				
		1	2	3	4	5
Group						
1	Size of group	11	8	15	16	18
	Average loss per member	82	84	100	110	
2	Size of group	5	9	8	11	
	Average loss per member	92	82	56	103	

Repeating the calculations from the previous question:

$$r = 2 \qquad n_1 = n_2 = 4$$

We can then calculate the necessary values of m_i and m:

$$m_1 = 11 + 8 + 15 + 16 = 50$$
$$m_2 = 5 + 9 + 8 + 11 = 33$$
$$m = 50 + 33 = 83$$

We need the X_{ij} values. These are the values in the table. This enables us to find $\bar{X}_i$:

$$\bar{X}_1 = \frac{1}{m_1}\sum_j m_{1j}X_{1j} = \frac{82\times 11 + 84\times 8 + 100\times 15 + 110\times 16}{50} = 96.68$$

$$\bar{X}_2 = \frac{1}{m_2}\sum_j m_{2j}X_{2j} = \frac{92\times 5 + 82\times 9 + 56\times 8 + 103\times 11}{33} = 84.2121$$

The overall mean is then:

$$\bar{X} = \frac{1}{m}\sum_i m_i \bar{X}_i$$
$$= \frac{1}{83}(50\times 96.68 + 33\times 84.2121) = 91.7229$$

This means that $\hat{\mu} = 91.7229$.

The formula for $\hat{\upsilon}$ is:

$$\hat{\upsilon} = \frac{1}{\sum_{i=1}^{r}(n_i - 1)}\sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{ij}(X_{ij} - \bar{X}_i)^2$$

So:

$$\hat{\upsilon} = \frac{1}{3+3}\begin{Bmatrix} 11(82-96.68)^2 + 8(84-96.68)^2 + 15(100-96.68)^2 \\ +16(110-96.68)^2 + 5(92-84.2121)^2 + 9(82-84.2121)^2 \\ +8(56-84.2121)^2 + 11(103-84.2121)^2 \end{Bmatrix}$$

$$= 2,876.3992$$

The formula for $\hat{a}$ is:

$$\hat{a} = \left(m - \frac{1}{m}\sum_{i=1}^{r} m_i^2 \right)^{-1} \left[\sum_{i=1}^{r} m_i (\bar{X}_i - \bar{X})^2 - \hat{\upsilon}(r-1) \right]$$

So:

$$\hat{a} = \frac{50(96.68-91.7229)^2 + 33(84.2121-91.7229)^2 - 2,876.3992}{\left(83 - \frac{1}{83}(50^2 + 33^2)\right)} = 5.3782$$

The estimate of k is:

$$\hat{k} = \frac{2,876.3992}{5.378} = 534.825$$

The credibility factor is then:

$$\hat{Z}_2 = \frac{33}{33 + 534.825} = 0.058$$

The credibility premium per individual is then:

$$\hat{Z}_2 \bar{X}_2 + (1 - \hat{Z}_2)\hat{\mu} = 0.058 \times 84.2121 + (1 - 0.058)91.7229 = 91.3$$

Solution 12.7

Both changes will reduce the sample variances for the two individual groups. This will reduce the value of $E[\upsilon(\Theta_i)]$. This means that the term we subtract in order to get the estimate of a will reduce, and the value of a is more likely to be positive.

Solution 12.8

From the question:

$$r = 1 \qquad n_1 = 4$$

We can then calculate the value of m_1:

$$m_1 = 25 + 22 + 29 + 30 = 106$$

The values in the table are the X_{1j} values. This enables us to find $\bar{X}_1$:

$$\bar{X}_1 = \frac{1}{m_1}\sum_j m_{1j}X_{1j} = \frac{90\times 25 + 96\times 22 + 100\times 29 + 99\times 30}{106} = 96.5283$$

From the question $\mu = 100$.

The formula for $\hat{\upsilon}$ is:

$$\hat{\upsilon}_i = \frac{\sum_{j=1}^{n_i} m_{ij}(X_{ij} - \bar{X}_i)^2}{n_i - 1}$$

So:

$$\begin{aligned}\hat{\upsilon} &= \frac{1}{3}\left\{25(90-96.5283)^2 + 22(96-96.5283)^2 + 29(100-96.5283)^2 + 30(99-96.5283)^2\right\} \\ &= 534.8050\end{aligned}$$

The formula for $\hat{a}$ is:

$$\hat{a}_i = (\bar{X}_i - \mu)^2 - \frac{\hat{\upsilon}_i}{m_i}$$

So:

$$\hat{a} = (96.5283 - 100)^2 - \frac{534.8050}{106} = 7.0074$$

The estimate of *k* is:

$$\hat{k} = \frac{534.8050}{7.0074} = 76.3205$$

The credibility factor is then:

$$\hat{Z}_2 = \frac{106}{106 + 76.3205} = 0.5814$$

The credibility premium per individual is then:

$$0.5814 \times 96.5283 + (1 - 0.5814)100 = 97.98$$

Solution 12.9

From the information given we have:

$$E\left[m_{ij}X_{ij} \mid \Theta_i\right] = m_{ij}\Theta_i \quad \Rightarrow \quad m_{ij}E\left[X_{ij} \mid \Theta_i\right] = m_{ij}\Theta_i \quad \Rightarrow \quad E\left[X_{ij} \mid \Theta_i\right] = \Theta_i$$

From the theory, $\mu(\Theta_i) = E\left[X_{ij} \mid \Theta_i\right]$, so:

$$\mu(\Theta_i) = \Theta_i$$

Also:

$$\text{var}(m_{ij}X_{ij} \mid \Theta_i) = m_{ij}\Theta_i(1-\Theta_i) \quad \Rightarrow \quad m_{ij}^2 \text{var}(X_{ij} \mid \Theta_i) = m_{ij}\Theta_i(1-\Theta_i)$$

$$\Rightarrow \quad \text{var}(X_{ij} \mid \Theta_i) = \frac{\Theta_i(1-\Theta_i)}{m_{ij}}$$

From the theory, $\dfrac{\upsilon(\Theta_i)}{m_{ij}} = \text{var}(X_{ij} \mid \Theta_i)$, so:

$$\upsilon(\Theta_i) = \Theta_i(1-\Theta_i)$$

This enables us to find μ, υ and a:

$$\mu = E\left[\mu(\Theta_i)\right] = E\left[\Theta_i\right]$$

$$\upsilon = E\left[\upsilon(\Theta_i)\right] = E\left[\Theta_i(1-\Theta_i)\right] = E\left[\Theta_i\right] - E\left[\Theta_i^2\right] = \mu - E\left[\Theta_i^2\right]$$

$$a = \text{var}(\mu(\Theta_i)) = \text{var}(\Theta_i) = E\left[\Theta_i^2\right] - \left[E\left[\Theta_i\right]\right]^2$$

Rewriting the last equation gives us:

$$a = \mu - \upsilon - \mu^2$$

Using values for the sample mean and variance given in a question in combination with the above equations enables us to find estimates for the structural parameters.

Solution 12.10

Using the formula from Solution 12.9:

$$\hat{a} = \hat{\mu} - \hat{\upsilon} - \hat{\mu}^2 = 0.12 - 0.03 - 0.12^2 = 0.0756$$

Solution 12.11

From this data we have:

$$r = 2 \qquad n = 4$$

We can calculate $\bar{X}_i$ and $\bar{X}$:

$$\bar{X}_A = \frac{450.9}{4} = 112.725 \qquad \bar{X}_B = \frac{1,367.3}{4} = 341.825$$

$$\bar{X} = \frac{1}{2}(112.725 + 341.825) = 227.275$$

$$\hat{\mu} = \bar{X} = 227.275$$

The required formula for $\hat{\upsilon}$ is:

$$\hat{\upsilon} = \frac{1}{r(n-1)} \sum_{i=1}^{r} \sum_{j=1}^{n} (X_{ij} - \bar{X}_i)^2$$

$$\hat{\upsilon} = \frac{1}{2(4-1)} \left\{ \begin{aligned} &(116.2 - 112.725)^2 + (111.9 - 112.725)^2 + (111.6 - 112.725)^2 + (111.2 - 112.725)^2 \\ &+ (345.1 - 341.825)^2 + (342.2 - 341.825)^2 + (341.1 - 341.825)^2 + (338.9 - 341.825)^2 \end{aligned} \right\} = 6.049167$$

The required formula for $\hat{a}$ is:

$$\hat{a} = \frac{1}{r-1} \sum_{i=1}^{r} (\bar{X}_i - \bar{X})^2 - \frac{\hat{\upsilon}}{n}$$

$$\begin{aligned} \hat{a} &= \frac{1}{1}\left\{(112.725 - 227.275)^2 + (341.825 - 227.275)^2\right\} - \frac{6.049167}{4} \\ &= 26,241.893 \end{aligned}$$

So: $\quad k = \dfrac{\upsilon}{a} = 0.0002305$ and: $\quad Z = 0.9999424$

So the premium for the coming year for Insurer A is:

$$CP_A = 0.99994 \times 112.725 + 0.00006 \times 227.275 = 112.732$$

Solution 12.12

For Insurer B, the corresponding premium is:

$$CP_B = 0.99994 \times 341.825 + 0.00006 \times 227.275 = 341.818$$

Solution 12.13

Calculating $\bar{X}_i$ and $\bar{X}$ as before:

$$\bar{X}_A = \frac{38.9}{4} = 9.725 \qquad \bar{X}_B = \frac{275.3}{4} = 68.825$$

$$\bar{X} = \frac{1}{2}(9.725 + 68.825) = 39.275$$

$$\hat{\mu} = \bar{X} = 39.275$$

The required formula for $\hat{\upsilon}$ is:

$$\hat{\upsilon} = \frac{1}{r(n-1)} \sum_{i=1}^{r} \sum_{j=1}^{n} (X_{ij} - \bar{X}_i)^2$$

$$\hat{\upsilon} = \frac{1}{2(4-1)} \left\{ \begin{array}{l} (4.2-9.725)^2 + (11.9-9.725)^2 + (3.6-9.725)^2 + (19.2-9.725)^2 \\ +(85.1-68.825)^2 + (60.2-68.825)^2 + (72.1-68.825)^2 + (57.9-68.825)^2 \end{array} \right\} = 105.31583$$

The required formula for $\hat{a}$ is:

$$\hat{a} = \frac{1}{r-1} \sum_{i=1}^{r} (\bar{X}_i - \bar{X})^2 - \frac{\hat{\upsilon}}{n}$$

$$\begin{aligned} \hat{a} &= \frac{1}{1}\left\{(9.725-39.275)^2 + (68.825-39.275)^2\right\} - \frac{105.31583}{4} \\ &= 1{,}720.076 \end{aligned}$$

So $k = 0.061227$ and $Z = 0.984924$. So the credibility premium for the coming year for Insurer A for the household portfolio is:

$$0.984924 \times 9.725 + 0.015076 \times 39.275 = 10.1705$$

Solution 12.14

For Insurer B, the corresponding figure is:

$$0.984924 \times 68.825 + 0.015076 \times 39.275 = 68.3795$$

Solution 12.15

Repeating the process for the auto portfolio figures, we have:

$$\bar{X}_A = \frac{412}{4} = 103 \qquad \bar{X}_B = \frac{1,092}{4} = 273$$

$$\bar{X} = \frac{1}{2}(103 + 273) = 188$$

$$\hat{\mu} = \bar{X} = 188$$

The required formula for $\hat{\upsilon}$ is:

$$\hat{\upsilon} = \frac{1}{r(n-1)} \sum_{i=1}^{r} \sum_{j=1}^{n} (X_{ij} - \bar{X}_i)^2$$

$$\hat{\upsilon} = \frac{1}{2(4-1)} \left\{ \begin{array}{l} (112-103)^2 + (100-103)^2 + (108-103)^2 + (92-103)^2 \\ + (260-273)^2 + (282-273)^2 + (269-273)^2 + (281-273)^2 \end{array} \right\} = 94.3333$$

The required formula for $\hat{a}$ is:

$$\hat{a} = \frac{1}{r-1} \sum_{i=1}^{r} (\bar{X}_i - \bar{X})^2 - \frac{\hat{\upsilon}}{n}$$

$$\hat{a} = \frac{1}{1}\left\{(103-188)^2 + (273-188)^2\right\} - \frac{94.3333}{4}$$
$$= 14,426.4167$$

So $k = 0.00653893$ and $Z = 0.9983679$. So the credibility premium for the coming year for Insurer A for the auto portfolio is:

$$0.9983679 \times 103 + 0.0016321 \times 188 = 103.139$$

Solution 12.16

Similarly, for Insurer B the premium is:

$$0.9983679 \times 273 + 0.0018321 \times 188 = 272.861$$

Solution 12.17

The total of the two individual credibility premiums for Insurer A for its two portfolios is:

$$10.171 + 103.139 = 113.31$$

The premium for Insurer A for the aggregate portfolio was 112.73. So the insurer pays a slightly higher premium if the portfolio is split in this way.

Solution 12.18

For Insurer B, the corresponding sum is:

$$68.380 + 272.861 = 341.24$$

The premium for Insurer B for the aggregate portfolio is 341.82. So Insurer B pays a very slightly lower premium if the portfolio is split (although the difference is pretty negligible).

Solution 12.19

All the credibility factors are high. These reflect:

(a) the large differences between the payments made by Insurer A and the payments made by Insurer B

(b) the relatively small variability within the experiences of both insurers.

Note that this is particularly the case in the aggregate model. Here the variability within the two different portfolios has been masked by the aggregation. (For example, in Insurer A's data, small payments of the household portfolio seem to be balanced by large payments on the auto portfolio, and *vice versa*. The net effect is that the aggregate portfolio payments are almost constant, whereas within the different portfolios there is much more variability.) This leads to a particularly high value for the credibility factor in the aggregate model.

Solution 12.20

If we carry out a Bühlmann-Straub analysis, we will be looking at the values of X_{ij} / m_{ij}. In this particular case, the higher payments made by Insurer B are (at least partly) a function of the higher policy numbers for Insurer B. Dividing through by the values of m_{ij} will bring the figures for the two insurers closer together. As a result, the variability between the two insurers will be reduced, and, other things being equal, we would expect this to result in lower credibility factors for both insurers if we try a Bühlmann-Straub model.