



Construction of Actuarial Models

Second Edition

by Mike Gauger

Published by BPP Professional Education

Solutions to practice questions - Chapter 14

Solution 14.1

$$\ln(X_1 \times \dots \times X_n) = \sum_{i=1}^n \ln(X_i) \sim N\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$$

$$\Rightarrow X_1 \times \dots \times X_n \sim \text{lognormal with parameters } \mu = \sum_{i=1}^n \mu_i, \sigma^2 = \sum_{i=1}^n \sigma_i^2$$

Solution 14.2

1. $e^{\mu+0.5\sigma^2} = E[X] = 40.44730 \Rightarrow \mu+0.5\sigma^2 = 3.700$
2. $e^{2\mu+2\sigma^2} = E[X^2] = (E[X])^2 + \text{var}(X) = 2,440.60163 \Rightarrow 2\mu+2\sigma^2 = 7.800$
3. $\sigma^2 = 7.800 - 2 \times 3.700 = 0.400, \mu = 3.500$

Solution 14.3

1. $z_{0.10} = 1.282 = 90^{\text{th}}$ percentile of $N(0,1)$
2. $\mu + 1.282\sigma = 3.5 + 1.282 \times \sqrt{0.4} = 4.31081 = 90^{\text{th}}$ percentile of $N(\mu, \sigma^2)$
3. $e^{\mu+1.282\sigma} = e^{4.31081} = 74.50 = 90^{\text{th}}$ percentile of $e^{N(\mu, \sigma^2)}$

Solution 14.4

$$\begin{aligned}
 \Pr\left(30 \leq e^{N(\mu, \sigma^2)} \leq 50\right) &= \Pr\left(\ln(30) \leq N(\mu, \sigma^2) \leq \ln(50)\right) \\
 &= \Phi\left(\frac{\ln(50) - \mu}{\sigma}\right) - \Phi\left(\frac{\ln(30) - \mu}{\sigma}\right) \\
 &= \Phi(0.651) - \Phi(-0.156) \\
 &= 0.7426 - 0.4379 = 0.3047
 \end{aligned}$$

Solution 14.5

$$\begin{aligned}
 l &= \frac{\ln(x) - \mu - \sigma^2}{\sigma} = \frac{\ln(38) - 3.5 - 0.40}{\sqrt{0.40}} = -0.415 \Rightarrow \Phi(l) = 0.3391 \\
 u &= \frac{\ln(x) - \mu}{\sigma} = \frac{\ln(38) - 3.5}{\sqrt{0.40}} = 0.218 \Rightarrow \Phi(u) = 0.5863 \\
 E[X | X < x] &= e^{\mu + 0.5\sigma^2} \times \frac{\Phi(l)}{\Phi(u)} = e^{3.7} \times \frac{0.3391}{0.5861} = 23.40
 \end{aligned}$$

Solution 14.6

$$\begin{aligned}
 l &= \frac{\ln(x) - \mu - \sigma^2}{\sigma} = \frac{\ln(45) - 3.5 - 0.40}{\sqrt{0.40}} = -0.148 \Rightarrow 1 - \Phi(l) = 0.5587 \\
 u &= \frac{\ln(x) - \mu}{\sigma} = \frac{\ln(45) - 3.5}{\sqrt{0.40}} = 0.485 \Rightarrow \Phi(u) = 0.6861 \\
 E[X | X > x] &= e^{\mu + 0.5\sigma^2} \times \frac{1 - \Phi(l)}{1 - \Phi(u)} = e^{3.7} \times \frac{0.5588}{0.3138} = 71.99
 \end{aligned}$$

Solution 14.7

$$\begin{aligned}
 104.08 &= E[S_t] = S_0 e^{(\alpha - \delta)t} = 100 e^{(\alpha - \delta)t} \Rightarrow e^{(\alpha - \delta)t} = 1.0408 \\
 E[S_{2.5t}] &= S_0 e^{(\alpha - \delta)2.5t} = 100 \left(e^{(\alpha - \delta)t}\right)^{2.5} = 110.51
 \end{aligned}$$

Solution 14.8

$$\begin{aligned}
 \alpha &= 0.10, \delta = 0, \sigma = 0.20, S_0 = 100 \\
 E[S_t] &= S_0 e^{(\alpha - \delta)t} \Rightarrow E[S_2] = 100 \times e^{0.20} = 122.14 \\
 \text{var}(S_t) &= S_0^2 e^{2(\alpha - \delta)t} \left(e^{\sigma^2 t} - 1\right) \Rightarrow \text{var}(S_2) = 100^2 \times e^{0.4} \times (e^{0.08} - 1) = 1,242.50
 \end{aligned}$$

Solution 14.9

Since returns on intervals of equal length are identically distributed, we have:

$$\frac{S_2}{S_1} = \frac{S_0 e^{R(0,2)}}{S_0 e^{R(0,1)}} = \frac{e^{R(0,1) + R(1,2)}}{e^{R(0,1)}} = e^{R(1,2)} \sim e^{R(0,1)}$$

So the distribution of $S_2 | S_1 = 105$ is the same as the distribution of S_1 when $S_0 = 105$:

$$\alpha = 0.10, \delta = 0, \sigma = 0.20, S_0 = 105$$

$$E[S_t] = S_0 e^{(\alpha - \delta)t} \Rightarrow E[S_1] = 105 \times e^{0.10} = 116.04$$

$$\text{var}(S_t) = S_0^2 e^{2(\alpha - \delta)t} (e^{\sigma^2 t} - 1) \Rightarrow \text{var}(S_1) = 105^2 \times e^{0.2} \times (e^{0.04} - 1) = 549.56$$

Solution 14.10

$$\alpha = 0.10, \delta = 0, \sigma = 0.20, S_0 = 100$$

$$\text{median of } S_t = S_0 e^{(\alpha - \delta - 0.5\sigma^2)t} \Rightarrow$$

$$\text{median of } S_1 = S_0 e^{(\alpha - \delta - 0.5\sigma^2)} = 100 \times e^{0.08} = 108.33$$

Solution 14.11

$$\alpha = 0.10, \delta = 0, \sigma = 0.20, S_0 = 100, K = 100, t = 0.5$$

$$1. \quad E[S_{0.5}] = S_0 e^{(\alpha - \delta)0.5} = 100 \times e^{0.05} = 105.127$$

$$2. \quad l = \frac{\ln(K/S_0) - (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}} = \frac{\ln(100/100) - 0.12 \times 0.5}{0.2 \times \sqrt{0.5}} = -0.424$$

$$\Rightarrow \Phi(l) = 0.3358$$

$$u = \frac{\ln(K/S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}} = \frac{\ln(100/100) - 0.08 \times 0.5}{0.2 \times \sqrt{0.5}} = -0.283$$

$$\Rightarrow \Phi(u) = 0.3886$$

$$3. \quad E[S_t | S_t < K] = E[S_t] \times \frac{\Phi(l)}{\Phi(u)}$$

$$\Rightarrow E[S_{0.5} | S_{0.5} < 100] = 90.84$$

Solution 14.12

Using the results of Solution 14.11, we have:

$$E[(K - S_t)_+] = K\Phi(u) - E[S_t] \Phi(l) = 100 \times 0.3886 - 105.127 \times 0.3358 = 3.575$$

Solution 14.13

Using the results of Solution 14.11, we have:

$$K - E[S_t | S_t < K] = K - E[S_t] \times \frac{\Phi(l)}{\Phi(u)} = 100 - 105.127 \times \frac{0.3358}{0.3886} = 9.20$$

Solution 14.14

$\alpha = 0.10$, $\delta = 0$, $\sigma = 0.20$, $S_0 = 100$, $K = 110$, $t = 0.5$

1. $E[S_{0.5}] = S_0 e^{(\alpha - \delta)0.5} = 100 \times e^{0.05} = 105.127$
 2. $l = \frac{\ln(K/S_0) - (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}} = \frac{\ln(110/100) - 0.12 \times 0.5}{0.2 \times \sqrt{0.5}} = 0.250$
 $\Rightarrow \Phi(l) = 0.5986$
 $u = \frac{\ln(K/S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}} = \frac{\ln(110/100) - 0.08 \times 0.5}{0.2 \times \sqrt{0.5}} = 0.391$
 $\Rightarrow \Phi(u) = 0.6521$
 3. $E[S_t | S_t > K] = E[S_t] \times \frac{1 - \Phi(l)}{1 - \Phi(u)}$
 $\Rightarrow E[S_{0.5} | S_{0.5} > 110] = 121.31$
-

Solution 14.15

Using the results of Solution 14.14, we have:

$$\begin{aligned} E[(S_t - K)_+] &= E[S_t] \Phi(-l) - K \Phi(-u) \\ &= 105.127 \times (1 - 0.5987) - 110 \times (1 - 0.6521) = 3.935 \end{aligned}$$

Solution 14.16

Using the results of Solution 14.14, we have:

$$\begin{aligned} E[S_t | S_t > K] - K &= E[S_t] \times \frac{\Phi(-l)}{\Phi(-u)} - K \\ &= 105.127 \times \frac{1 - 0.5987}{1 - 0.6521} - 110 = 11.26 \end{aligned}$$

Solution 14.17

$$\text{call payoff} = (S_t - K)_+ = \begin{cases} 0 & \text{if } S_t \leq K \\ S_t - K & \text{if } S_t > K \end{cases}$$

$$\text{put payoff} = (K - S_t)_+ = \begin{cases} K - S_t & \text{if } S_t \leq K \\ 0 & \text{if } S_t > K \end{cases}$$

$$\Rightarrow \text{call payoff} - \text{put payoff} = S_t - K$$

Solution 14.18

Using the results of Solutions 14.12 and 14.17, we have:

$$\text{Solution 14.17: } E[\text{call payoff}] - \underbrace{E[\text{put payoff}]}_{= 3.575 \text{ (see Solution 14.12)}} = \underbrace{E[S_t]}_{105.127} - \underbrace{K}_{100}$$

$$\Rightarrow E[\text{call payoff}] = 8.702$$

Solution 14.19

From the given weekly price data we can determine the corresponding weekly returns:

weekly prices: 100, 102, 106, 98, 100, 97, 100, 103, 108, 102 \Rightarrow

$$r_1 = \ln(102/100) = 0.0198 \quad r_2 = \ln(106/102) = 0.0385$$

$$r_3 = \ln(98/106) = -0.0785 \quad r_4 = \ln(100/98) = 0.0202$$

$$r_5 = \ln(97/100) = -0.0305 \quad r_6 = \ln(100/97) = 0.0305$$

$$r_7 = \ln(103/100) = 0.0296 \quad r_8 = \ln(108/103) = 0.0474$$

$$r_9 = \ln(102/108) = -0.0572$$

Now determine the estimates:

$h = 1/52$ (week is a 52nd of a year)

$$\bar{r} = \frac{1}{n}(r_1 + \dots + r_n) = \frac{1}{9}(0.0198) = 0.0022$$

$$\hat{\sigma}^2 = \frac{1}{h \times (n-1)} \sum_{i=1}^n (r_i - \bar{r})^2 = \frac{1}{h \times (n-1)} \left(\sum_{i=1}^n r_i^2 - n\bar{r}^2 \right) = \frac{1}{(1/52) \times 8} (0.01668 - 9 \times 0.0022^2) = 0.10814$$

$$\Rightarrow \hat{\sigma} = 0.3288$$

Solution 14.20

$$\hat{\alpha} = \frac{\bar{r}}{h} + 0.5\hat{\sigma}^2 = 0.1685 \quad (\text{See Solution 14.19})$$