



Construction of Actuarial Models

Third Edition

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Solutions to practice questions – Chapter 15

Solution 15.1

$$\begin{aligned}
 C - P &= e^{-rt} \left(E[(S_t - K)_+] - E[(K - S_t)_+] \right) \\
 &= e^{-rt} \times E[(S_t - K)_+ - (K - S_t)_+] \\
 &= e^{-rt} \times E[S_t - K] \quad (\text{see the Solution to Question 14.17}) \\
 &= e^{-rt} \times (S_0 e^{(r-\delta)t} - K)
 \end{aligned}$$

Solution 15.2

$$\frac{\partial \left(e^{-rt} \times (S_0 e^{(r-\delta)t} - K) \right)}{\partial K} = -e^{-rt}$$

Interpretation: a change of ΔK in the strike price causes the difference between the call and put prices to decrease by $e^{-rt} \Delta K$.

Solution 15.3

$$E[S_{0.5}] = S_0 e^{(r-\delta)0.5} = 100 e^{0.08 \times 0.5} = 104.0811$$

$$\begin{aligned}
 l &= \frac{\ln(K/S_0) - (r - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}} \\
 &= \frac{\ln(100/100) - (0.08 + 0.5 \times 0.20^2) \times 0.50}{0.20 \times \sqrt{0.50}} = -0.3536
 \end{aligned}$$

$$\Rightarrow \Phi(-l) = \Phi(0.3536) = 0.6382$$

$$u = \frac{\ln((K/S_0)) - (r - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}$$

$$= \frac{\ln(100/100) - (0.08 - 0.5 \times 0.20^2) \times 0.50}{0.20 \times \sqrt{0.50}} = -0.2121$$

$$\Rightarrow \Phi(-u) = \Phi(0.2121) = 0.5840$$

$$C = e^{-rt} (E[S_t] \Phi(-l) - K \Phi(-u))$$

$$= e^{-0.08 \times 0.5} (104.0811 \times 0.6382 - 100 \times 0.5840) = 7.71$$

Solution 15.4

We can use some of the same calculations as in Solution 15.3:

$$E[S_{0.5}] = 104.0811$$

$$l = -0.3536 \Rightarrow \Phi(l) = 1 - \Phi(0.3536) = 0.3618$$

$$u = -0.2121 \Rightarrow \Phi(u) = 1 - \Phi(0.2121) = 0.4160$$

$$P = e^{-rt} (K \Phi(u) - E[S_t] \Phi(l))$$

$$= e^{-0.08 \times 0.50} (100 \times 0.4160 - 108.0811 \times 0.3618)$$

$$= 3.79$$

Solution 15.5

According to Solution 15.1 the difference is:

$$C - P = e^{-rt} \times (S_0 e^{(r-\delta)t} - K) = e^{-0.04} (104.0811 - 100) = 3.92$$

$$\frac{7.71}{\text{Solution 15.3}} - \frac{3.79}{\text{Solution 15.4}} = 3.92$$

Solution 15.6

The inversion method results in a simulated value as follows:

$$S_t = S_0 e^{(r-\delta-0.5\sigma^2)t} \times e^{\sigma\sqrt{t} \Phi^{-1}(u)}$$

$$= 100 e^{(0.08-0.02-0.5 \times 0.04) \times 2} \times e^{0.2 \times \sqrt{2} \times \Phi^{-1}(0.1401)}$$

$$= 100 e^{0.08} \times e^{-0.3055} \quad (\text{since } \Phi^{-1}(0.1401) = -1.08)$$

$$= 79.81$$

Solution 15.7

The simulated value of $S_{0.50}$ is:

$$\begin{aligned} S_t &= S_0 e^{(r-\delta-0.5\sigma^2)t} \times e^{\sigma\sqrt{t} \Phi^{-1}(u)} \\ &= 100 e^{(0.08-0.02-0.5\times 0.04)\times 0.50} \times e^{0.20\times\sqrt{0.5}\times\Phi^{-1}(0.7054)} \\ &= 100 e^{0.02} \times e^{0.0764} = 110.12 \end{aligned}$$

The present value of the call option payoff is:

$$e^{-rt} \times (S_t - K)_+ = e^{-0.08 \times 0.50} \times (S_{0.50} - 102)_+ = 7.80$$

Solution 15.8

Using results of Solution 15.7, the present value of the put option payoff is:

$$e^{-rt} \times (K - S_t)_+ = e^{-0.08 \times 0.50} \times (102 - 110.12)_+ = 0 \quad (\text{Solution 15.7: } S_t = 110.12)$$

Solution 15.9

The starting value and recursion formula for the stock price path are:

$$\begin{aligned} S_0 &= 50, \quad h = (1/3)/4 = 1/12 = 0.0833 \\ S_{hi} &= S_{h(i-1)} e^{(r-\delta-0.5\sigma^2)h} \times e^{\sigma\sqrt{h} z_i} \\ &= S_{h(i-1)} \times 1.00501 \times e^{0.05774 z_i} \quad \text{where } z_i = \Phi^{-1}(u_i) \end{aligned}$$

From the standard normal probability table, we have the following path of simulated stock prices:

$$\begin{aligned} \Phi^{-1}(0.9192) &= 1.40 \Rightarrow S_h = 50 \times 1.00501 \times e^{0.05774 \times 1.40} = 54.4810 \\ \Phi^{-1}(0.4840) &= -0.04 \Rightarrow S_{2h} = 54.4810 \times 1.00501 \times e^{0.05774 \times (-0.04)} = 54.6278 \\ \Phi^{-1}(0.6141) &= 0.29 \Rightarrow S_{3h} = 54.6278 \times 1.00501 \times e^{0.05774 \times 0.29} = 55.8286 \\ \Phi^{-1}(0.2420) &= -0.70 \Rightarrow S_{4h} = 55.8286 \times 1.00501 \times e^{0.05774 \times (-0.70)} = 53.8860 \\ \Rightarrow \bar{S}_t &= \frac{S_h + S_{2h} + S_{3h} + S_{4h}}{4} = 54.7059 \\ \Rightarrow Y &= e^{-rt} \left(\bar{S}_t - K \right)_+ = e^{-0.08 \times (1/3)} \times (54.7059 - 52)_+ = 2.63 \end{aligned}$$

Solution 15.10

$$z_1 = \Phi^{-1}(0.5793) = 0.20$$

$$z_2 = \Phi^{-1}(0.3300) = -0.44$$

$$z_S = z_1 = 0.20$$

$$z_T = \rho z_1 + \sqrt{1 - \rho^2} z_2 = 0.20 \times 0.20 + \sqrt{0.96} \times (-0.44) = -0.3911$$

$$\begin{aligned} R_S(0, t) &= (\alpha_S - \delta_S - 0.5\sigma_S^2)t + \sigma_S \sqrt{t} z_S = \\ &= (0.07 - 0.5 \times 0.16) \times 1 + 0.4 \sqrt{1} \times 0.20 = 0.07 \end{aligned}$$

$$\Rightarrow S_t = S_0 e^{R_S(0, t)} = 100 e^{0.07} = 107.25$$

$$\begin{aligned} R_T(0, t) &= (\alpha_T - \delta_T - 0.5\sigma_T^2)t + \sigma_T \sqrt{t} z_T \\ &= (0.06 - 0.5 \times 0.04) \times 1 + 0.2 \sqrt{1} \times (-0.3911) = \end{aligned}$$

$$\Rightarrow T_t = T_0 e^{R_T(0, t)} = 90 e^{-0.0382} = 86.62$$

Solution 15.11

1. $z = \Phi^{-1}(u_1) = \Phi^{-1}(0.6772) = 0.46$
2. Use the random number $u_2 = 0.8238$ to simulate $n = N(1) = \text{Poisson mean } 0.10$
 $\Pr(N(1)=0) = e^{-0.20} = 0.81873$
 $\Pr(N(1) \leq 1) = e^{-0.20} (1 + 0.20) = 0.98248$
 $0.81873 < u_2 < 0.98248 \Rightarrow n = 1$
3. Use random number u_3 to simulate $w_1 = \Phi^{-1}(u_3) = \Phi^{-1}(0.3446) = -0.40$
4. The simulated stock price at time 1 is thus:

$$k = e^{\alpha J} - 1 = -0.07688$$

$$\begin{aligned} S_t &= S_0 e^{(\alpha - \delta - \lambda k - 0.5\sigma^2)t + \sigma \sqrt{t} z + (\alpha_J - 0.5\sigma_J^2)n + \sigma_J(w_1 + \dots + w_n)} \\ &= 100 e^{(0.09 - 0 + 0.20 \times 0.07688 - 0.5 \times 0.09) + 0.3 \times 0.46 + (-0.08 - 0.5 \times 0.0016) + 0.04 \times (-0.40)} \\ &= 100 e^{0.1016} = 110.69 \end{aligned}$$

Solution 15.12

$$\begin{aligned}
Y_t &= e^{(\alpha-0.5\sigma^2)t + \sigma\sqrt{t}Z} \times e^{\beta N(t)} \\
&= e^{(\alpha-0.5\sigma^2)t} \times e^{N(0, \sigma^2 t)} \times e^{\beta N(t)} \\
\Rightarrow E[Y_t] &= e^{(\alpha-0.5\sigma^2)t} \times e^{0.5\sigma^2 t} \times M_{N(t)}(\beta) \\
&= e^{\alpha t} \times e^{\lambda t(e^\beta - 1)} = e^{\alpha t} \times e^{\lambda t k} \\
&= e^{(\alpha + \lambda k)t}
\end{aligned}$$

$$\begin{aligned}
Y_t^2 &= e^{(\alpha-0.5\sigma^2)2t + 2\sigma\sqrt{t}Z} \times e^{2\beta N(t)} \\
&= e^{(\alpha-0.5\sigma^2)2t} \times e^{N(0, 4\sigma^2 t)} \times e^{2\beta N(t)} \\
\Rightarrow E[Y_t^2] &= e^{(\alpha-0.5\sigma^2)2t} \times e^{2\sigma^2 t} \times M_{N(t)}(2\beta) \\
&= e^{(2\alpha + \sigma^2)t} \times e^{\lambda t(e^{2\beta} - 1)} = e^{(2\alpha + \sigma^2)t} \times e^{\lambda t(k^2 + 2k)} \\
&= e^{(2\alpha + \sigma^2 + 2\lambda k + \lambda k^2)t}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \text{var}(Y_t) &= e^{(2\alpha + \sigma^2 + 2\lambda k + \lambda k^2)t} - \left(e^{(\alpha + \lambda k)t} \right)^2 \\
&= e^{(2\alpha + 2\lambda k)t} \left(e^{(\sigma^2 + \lambda k^2)t} - 1 \right)
\end{aligned}$$