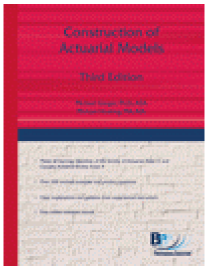

# Construction of Actuarial Models

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## *Solutions to practice questions – Chapter 16*

### *Solution 16.1*

The sample mean is:

$$\bar{x} = \frac{\sum x_i}{10} = \frac{1,266}{10} = 126.6$$

We now need the sum of the squared deviations for those losses that exceed 126.6:

$$(180-126.6)^2 + (270-126.6)^2 + (560-126.6)^2 = 211,250.68$$

So the upside semi-variance is:

$$\frac{211,250.68}{10} = 21,125$$

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### *Solution 16.2*

The threshold upside semi-variance is:

$$\frac{80^2 + 170^2 + 460^2}{10} = 24,690$$

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## *Solution 16.3*

Using the upper 5% point of the normal distribution, we have:

$$Q_{0.95} = 500 + 1.6449\sqrt{100} = 516.449$$

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## *Solution 16.4*

The normal distribution will have a thinner right hand tail than the corresponding lognormal distribution.

For large claim amounts, the lognormal curve will lie above the corresponding normal curve.

So if we want to contain an area to the right equal to 5%, we will need to move our risk measure to the right, *ie* it should be a little bigger. Let's now check this out.

First we need the parameters of the lognormal distribution:

$$e^{\mu+½\sigma^2} = 500 \qquad e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right) = 100$$

Solving these simultaneous equations, we find that $\mu = 6.214408$ and $\sigma^2 = 0.00039992$.

We now need the 95th percentile of this distribution:

$$\Pr\left(\log N(\mu,\sigma^2) < Q\right) = \Pr\left(N(\mu,\sigma^2) < \log Q\right) = \Pr\left(N(0,1) < \frac{\log Q - \mu}{\sigma}\right) = 0.95$$

This gives us the equation:

$$\frac{\log Q - 6.214408}{\sqrt{0.00039992}} = 1.6449 \qquad \Rightarrow \qquad Q = 516.62$$

As we anticipated, we have a slightly greater risk measure.

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## *Solution 16.5*

We need to solve the equation:

$$F(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha} = 0.95$$

Substituting in $\theta = 50$ and $\alpha = 2.5$ and rearranging, we find that:

$$x_{0.95} = \frac{50}{0.05^{0.4}} - 50 = 115.723$$

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## *Solution 16.6*

Using the fact that $X - x_\alpha \mid X > x_\alpha$ is also Pareto, but with parameters $\theta' = 50 + 115.723$ and $\alpha = 2.5$, we see that the value of the 95% CTE risk measure in this case is:

$$x_\alpha + \frac{\theta'}{\alpha - 1} = 115.723 + \frac{50 + 115.723}{1.5} = 226.205$$

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## *Solution 16.7*

First we need the VaR risk measure. Using the distribution function of the exponential distribution:

$$F(x) = 0.98 = 1 - e^{-x/500} \quad \Rightarrow \quad x_\alpha = -500 \log 0.02 = 1,956.01$$

The exponential distribution also has a useful conditional distribution result. If $X$ has an exponential distribution with mean $\theta$, then $X - x_\alpha \mid X > x_\alpha$ also has an exponential distribution with the same mean (this is the lack of memory property of the exponential distribution). So the CTE risk measure is:

$$1,956.01 + 500 = 2,456.01$$

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## *Solution 16.8*

First, we see that the CDF of the loss distribution has a discontinuity at $x = 50$. The distribution jumps up from $1 - 0.082085 = 0.917915$ to 1 at that point.

So $Q_{0.95}$ and $Q_{0.99}$ are both equal to 50.

But the CTE risk measures are the values of the conditional expectation, given that the loss is greater than or equal to 50. But in this case the loss must be exactly 50, since it can never exceed 50. So both CTE risk measures are equal to 50.

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## *Solution 16.9*

The distribution function for the Weibull distribution is:

$$F(x) = 1 - e^{-(x/\theta)^\tau}$$

We need to solve the equation:

$$1 - e^{-(x/500)^2} = 0.95 \quad \Rightarrow \quad x = \sqrt{-500^2 \log 0.05} = 865.409$$

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## *Solution 16.10*

The undistorted probability of loss is:

$$S(1{,}000) = e^{-(1{,}000/\theta)^{\tau}} = e^{-(2)^2} = e^{-4} = 0.01832$$

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## *Solution 16.11*

Using a proportional hazards transform with $\kappa = 2$, we obtain a distorted survival function of:

$$g\left(S(x)\right) = \left[e^{-(x/\theta)^{\tau}}\right]^{½} = e^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{\tau}} = e^{-\left(\frac{x}{2^{1/\tau}\theta}\right)^{\tau}}$$

We see that this is the survival function of a new Weibull distribution, with parameters $\tau = 2$ and $\theta = 2^{½} \times 500 = 707.1068$.

So the new survival probability is now:

$$S(1{,}000) = e^{-\left(\frac{1{,}000}{707.1068}\right)^2} = e^{-2} = 0.1353$$

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## *Solution 16.12*

Using a dual power transform with $\kappa = 2$, we have:

$$g\left(S(x)\right) = 1 - \left[1 - e^{-(x/\theta)^{\tau}}\right]^2 = 1 - \left[1 - e^{-4}\right]^2 = 0.036296$$

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## *Solution 16.13*

Using Wang's transform with $\kappa = 1$, we have:

$$\begin{aligned} g\left(S(1{,}000)\right) &= \Phi\left[\Phi^{-1}\left(S(1{,}000)\right) + 1\right] = \Phi\left[\Phi^{-1}\left(0.01832\right) + 1\right] \\ &= \Phi\left(-2.09 + 1\right) = \Phi(-1.09) = 0.1379 \end{aligned}$$

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## *Solution 16.14*

First we find the probability that $X = 0$:

$$\Pr(X = 0) = \Pr(S \geq 1{,}000) = \Pr\left(1{,}000e^{0.04+0.2Z} \geq 1{,}000\right) = \Pr\left(0.04 + 0.2Z \geq 0\right)$$
$$= \Pr\left(N(0.04, 0.2^2) \geq 0\right) = \Pr\left(N(0,1) \geq -\frac{0.04}{0.2}\right) = 1 - \Phi(-0.2) = 0.5793$$

Since $0.9 > 0.5793$, the 90th percentile of the distribution of $X$ lies in the continuous part of the distribution.

We want:

$$0.90 = \Pr(X \leq x_{0.90}) = \Pr\left(1{,}000 - S \leq x_{0.90}\right) = \Pr\left(1{,}000 - 1{,}000e^{0.04+0.2Z} \leq x_{0.90}\right)$$
$$= \Pr\left(e^{0.04+0.2Z} \geq \frac{1{,}000 - x_{0.90}}{1{,}000}\right)$$

But $e^{0.04+0.2Z}$ has a lognormal distribution with $\mu = 0.04$ and $\sigma = 0.2$. So we want the 10th percentile of this lognormal distribution.

The 10th percentile of $N(0,1)$ is $-1.282$. The 10th percentile of $N(0.04, 0.2^2)$ is $0.04 - (1.282 \times 0.2) = -0.2164$. The 10th percentile of the corresponding lognormal distribution is $e^{-0.2164} = 0.80541$.

So we now have the equation:

$$\frac{1{,}000 - x_{0.90}}{1{,}000} = 0.80541$$

which gives us $x_{0.90} = 194.59$.

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## *Solution 16.15*

We now have:

$$X = (S - 1{,}000)_+$$

where $S$ is defined as before. Again we find the probability that $X = 0$:

$$\Pr(X = 0) = \Pr(S \le 1{,}000) = 1 - 0.5793 = 0.4207$$

So again the 90th percentile of the distribution of $X$ lies in the continuous part of the distribution.

We want:

$$0.90 = \Pr(X \le x_{0.90}) = \Pr\left(S - 1{,}000 \le x_{0.90}\right) = \Pr\left(1{,}000e^{0.04+0.2Z} - 1{,}000 \le x_{0.90}\right)$$
$$= \Pr\left(e^{0.04+0.2Z} \le \frac{x_{0.90} + 1{,}000}{1{,}000}\right)$$

But $e^{0.04+0.2Z}$ has a lognormal distribution with $\mu = 0.04$ and $\sigma = 0.2$. So we want the 90th percentile of this lognormal distribution.

The 90th percentile of $N(0,1)$ is +1.282. The 90th percentile of $N(0.04, 0.2^2)$ is $0.04 + (1.282 \times 0.2) = 0.2964$. The 90th percentile of the corresponding lognormal distribution is $e^{+0.2964} = 1.345$.

So we now have the equation:

$$\frac{x_{0.90} + 1{,}000}{1{,}000} = 1.345 \quad \Rightarrow \quad x_{0.90} = 345$$

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## *Solution 16.16*

First we find the relevant VaR risk measures. These are $Q_{0.9} = 10$, $Q_{0.95} = 100$ and $Q_{0.99} = 500$.

We now look at the outcomes for the largest 10% of probability. Of the largest 10% of probability, 0.02 of it is at 10, 0.06 is at 100, 0.016 is at 500 and 0.004 is at 1,000. Factoring these up by dividing by 0.1, we obtain the corresponding conditional distribution:

$$\Pr(X = 10) = 0.2 \qquad \Pr(X = 100) = 0.6 \qquad \Pr(X = 500) = 0.16 \qquad \Pr(X = 1{,}000) = 0.04$$

The mean of this conditional distribution is :

$$10 \times 0.2 + 100 \times 0.6 + 500 \times 0.16 + 1{,}000 \times 0.04 = 182.$$

This is the 90% CTE risk measure.

 

Alternatively we can use the formula

$$\text{CTE}_\alpha(X) = \frac{(\beta-\alpha)x_\alpha + (1-\beta)E\left[X \mid X > x_\alpha\right]}{1-\alpha}$$

with $\beta = 0.92$, $\alpha = 0.9$, and:

$$E\left[X \mid X > 10\right] = 100 \times \frac{.06}{.08} + 500 \times \frac{.016}{.08} + 1,000 \times \frac{.004}{.08} = 225$$

So now we get:

$$CTE_{0.90} = \frac{(0.92-0.9)10 + (1-0.92)225}{0.1} = 182$$

This gives the same answer as the more intuitive approach.

You should be able to obtain (using either method) the corresponding results:

$$CTE_{0.95} = 300 \qquad CTE_{0.99} = 700$$

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## *Solution 16.17*

First we find the corresponding quantile risk measures. By accumulating the probability, we can see that $x_{0.9} = 90$ and also $x_{0.95} = 90$. Since the final 2% of probability is uniformly spread over the interval $(90,100)$, we see that $x_{0.99} = 95$.

Again we condition on that part of the distribution which lies above the 90th percentile. The conditional distribution is now a mixed distribution, 0.08 of probability at $x = 90$ and 0.02 spread uniformly over $(90,100)$. Factoring these up, we have:

$$\Pr(X = 90) = 0.8$$
$$f(x) = 0.02 \qquad 90 < x \le 100$$

[Note that when the 0.02 of probability is originally spread over $(90,100)$, it produces a PDF of 0.002. When this is factored up, it goes back to being 0.02.]

The mean of this distribution is:

$$0.8 \times 90 + 0.2 \times 95 = 91$$

Alternatively, we can use:

$$\text{CTE}_\alpha(X) = \frac{(\beta-\alpha)x_\alpha + (1-\beta)E\left[X \mid X > x_\alpha\right]}{1-\alpha} = \frac{(0.98-0.9)90 + 0.02(95)}{0.1} = 91$$

using $\alpha = 0.9$, $\beta = 0.98$, and $E\left[X \mid X > x_\alpha\right] = 95$.

Similarly, for $\alpha = 0.95$, we obtain a conditional distribution for which:

$$\Pr(X = 90) = 0.6$$
$$f(x) = 0.04 \qquad 90 < x \le 100$$

The mean of this conditional distribution is:

$$0.6 \times 90 + 0.4 \times 95 = 92$$

Again, the weighted average formula would also give us the correct answer.

If $\alpha = 0.99$, the 99th percentile does not lie at a discontinuity on the CDF. So we can just use $E[X \mid X > 95]$, which is the mean of a uniform distribution on the interval $(95,100)$. The answer is 97.5.

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## *Solution 16.18*

We first need to find the lognormal parameters. Solving the simultaneous equations:

$$e^{\mu+\frac{1}{2}\sigma^2} = 12.2 \qquad \text{and:} \qquad e^{2\mu+\sigma^2}\left[e^{\sigma^2} - 1\right] = 256$$

we find that $\mu = 2$ and $\sigma = 1$.

The upside semi-variance is given by:

$$\sigma^2_{SVU} = \int_m^\infty (x-m)^2 f(x)\,dx = \int_m^\infty x^2 f(x)dx - 2m\int_m^\infty xf(x)dx + m^2\int_m^\infty f(x)dx$$

using $m$ for the mean of the lognormal distribution, to avoid confusion with the parameter $\mu$.

Using the result given in the question with $k = 2$, $k = 1$ and $k = 0$ in turn, we find that the upside semi-variance is:

$$e^{2\mu+2\sigma^2}\left[1 - \Phi\left(\frac{\log m - \mu}{\sigma} - 2\sigma\right)\right] - 2m\,e^{\mu+\frac{1}{2}\sigma^2}\left[1 - \Phi\left(\frac{\log m - \mu}{\sigma} - \sigma\right)\right]$$
$$+ m^2\left[1 - \Phi\left(\frac{\log m - \mu}{\sigma}\right)\right]$$

Substituting in the values $m = 12.2$, $\mu = 2$, $\sigma = 1$, we obtain a value of 216.84.

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## *Solution 16.19*

Th 99th percentile will be approximated by the 990th sample value, which will be 450.31.

[Alternatively, you could use 450.36, or a weighted average of these two figures].

The 99% CTE estimate is the mean of the figures in the last row of the table:

$$\bar{x} = \frac{1}{10}\left(450.36 + \cdots + 767.20\right) = 556.52$$

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## *Solution 16.20*

The number of sample values below the 990th value will have a binomial distribution with parameters $n = 1{,}000$ and $\theta = 0.99$. We will approximate this using a normal distribution with mean 990 and variance 9.9.

The distribution of the corresponding sample proportion will be normal with mean 0.990 and variance $9.9 \times 10^{-6}$.

So the confidence interval will be:

$$0.99 \pm 1.96\sqrt{9.9 \times 10^{-6}} = (0.98383, 0.99617)$$

So the interval in terms of sample values should run from the 983rd to the 997th value inclusive. This gives us a confidence interval of (405.47, 580.22).

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