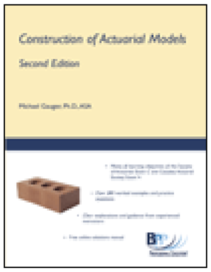

Construction of Actuarial Models

Second Edition

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Solutions to practice questions – Chapter 2

Solution 2.1

We need only use the Pareto moment formulas given in Section 2.2 in the Pareto summary:

$$E\left[X^k\right] = \frac{\theta^k \, k!}{(\alpha-1)(\alpha-2)\cdots(\alpha-k)} \quad \text{if} \quad k < \alpha$$

$$\alpha = 3\,,\ \theta = 1{,}000 \ \Rightarrow\ E[X] = \frac{1{,}000}{2} = 500\ ,\ E\left[X^2\right] = \frac{1{,}000^2 \times 2!}{2\times 1} = 1{,}000{,}000$$

$$\Rightarrow\ \operatorname{var}(X) = 1{,}000{,}000 - 500^2 = 750{,}000$$

Solution 2.2

The payment per loss is given by:

$$Y = \begin{cases} X & \text{if } X \le 1{,}000 \\ 1{,}000 & \text{if } X > 1{,}000 \end{cases}$$

Since Y is a function of X it is straightforward to compute its expected value:

$$f(x) = \frac{3\times 1{,}000^3}{(x+1{,}000)^4} \quad \text{for } x > 0$$

$$E[Y] = \int_0^\infty y\, f_X(x)\,dx = \int_0^{1{,}000} \frac{3\times 1{,}000^3\, x}{(x+1{,}000)^4}dx + \int_{1{,}000}^\infty \frac{3\times 1{,}000^4}{(x+1{,}000)^4}dx$$

$$= \int_0^{1{,}000} \frac{3\times 1{,}000^3\,(x + 1{,}000 - 1{,}000)}{(x+1{,}000)^4}dx + \int_{1{,}000}^\infty \frac{3\times 1{,}000^4}{(x+1{,}000)^4}dx$$

$$= \int_0^{1{,}000} \frac{3\times 1{,}000^3}{(x+1{,}000)^3}dx - \int_0^{1{,}000} \frac{3\times 1{,}000^4}{(x+1{,}000)^4}dx + \int_{1{,}000}^\infty \frac{3\times 1{,}000^4}{(x+1{,}000)^4}dx$$

$$= \left(-\frac{3\times 1{,}000^3}{2(x+1{,}000)^2}\Bigg|_0^{1{,}000}\right) - \left(-\frac{3\times 1{,}000^4}{3(x+1{,}000)^3}\Bigg|_0^{1{,}000}\right) + \left(-\frac{3\times 1{,}000^4}{3(x+1{,}000)^3}\Bigg|_{1{,}000}^{\infty}\right)$$

$$= 1{,}125 - 875 + 125 = 375$$

Solution 2.3

Note first that $E[X]=e^{\mu+\sigma^2/2}$ according to the moment formula in the lognormal summary. Hence we have:

$$\Pr\left(X > e^{\mu+\sigma^2/2}\right) = \Pr\left(\ln(X) > \mu + 0.5\sigma^2\right) = \Pr\left(N\left(\mu,\sigma^2\right) > \mu + 0.5\sigma^2\right)$$

$$= \Pr\left(N(0,1) > \frac{\mu + 0.5\sigma^2 - \mu}{\sigma}\right) = 1-\Phi(0.5\sigma) = 1-\Phi(0.5)$$

Solution 2.4

For both policies #1 and #2, the policyholder retains the entire loss amount equal to 75 since there is no reimbursement for a loss less than the deductible. For a loss equal to 150, the owner of policy #1 will receive a reimbursement of $150-100=50$. On the other hand, the owner of policy #2 will be reimbursed for the full 150 loss. Remember that when you have a franchise deductible, then any loss exceeding the deductible is fully reimbursed.

Solution 2.5

The pdf for the loss amount X is: $f_X(x) = 0.001$ for $0 < x < 1{,}000$.

If there is an **ordinary deductible** of 100 per loss, then from the ordinary deductible summary in Section 2.3, we have the following:

$$f_y(y) = \begin{cases} F_X(100) & \text{if } y = 0 \quad \text{(the discrete part)} \\ f_X(y+100) & \text{if } y > 0 \quad \text{(the continuous part)} \end{cases}$$

$$= \begin{cases} 100/1{,}000 & \text{if } y = 0 \quad \text{(the discrete part)} \\ 0.001 & \text{if } 0 < y < 900 \quad \text{(the continuous part)} \end{cases}$$

If there is a **franchise deductible** of 100 per loss, then from the franchise deductible summary in Section 2.3, we have the following:

$$f_Y(y) = \begin{cases} F_X(100) & \text{if } y = 0 \quad \text{(discrete part)} \\ f_X(y) & \text{if } y > 100 \quad \text{(continuous part)} \end{cases}$$

$$= \begin{cases} 100/1{,}000 & \text{if } y = 0 \quad \text{(discrete part)} \\ 0.001 & \text{if } 100 < y < 1{,}000 \quad \text{(continuous part)} \end{cases}$$

Solution 2.6

Suppose that there is an **ordinary deductible** of 100 per loss. From the ordinary deductible summary found in Section 2.3, we have:

$$\begin{aligned} f_Z(z) &= \frac{f_X(z+d)}{s_X(d)} = \frac{f_X(z+100)}{s_X(100)} \quad \text{for } z>0 \\ &= \frac{0.001}{900/1{,}000} \quad \text{for } 0<z<900 \\ &= \frac{1}{900} \quad \text{for } 0<z<900 \quad \text{(a uniform distribution on } [0,900]) \end{aligned}$$

Suppose that there is a **franchise deductible** of 100 per loss. From the franchise deductible summary found in Section 2.3, we have:

$$\begin{aligned} f_Z(z) &= \frac{f_X(z)}{s_X(d)} \quad \text{for } z>d \\ &= \frac{f_X(x)}{s_X(100)} \quad \text{for } z>100 \\ &= \frac{0.001}{900} \quad \text{for } 100<z<1{,}000 \quad \left(\text{a uniform distribution on } [100,1000]\right) \end{aligned}$$

Solution 2.7

Payment events are the same as loss events when the only loss-limiting feature is a coinsurance factor. So the expected value and variance of the insurance payment per loss (*ie* the claim payment related to a single loss) are determined as follows:

$$Y=0.85X \;\Rightarrow\; E\left[Y^k\right]=0.85^k E\left[X^k\right]=0.85^k \times \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}$$

$$E[Y]=0.85\times 500=425 \;,\quad E\left[Y^2\right]=0.85^2\times 1{,}000{,}000=722{,}500$$

$$\text{var}(Y)=722{,}500-425^2=541{,}875$$

It would also be possible to use the fact that Y follows a 2-parameter Pareto distribution with the same $\alpha=3$ and with $\theta^*=0.85\theta=850$ since θ is a scale parameter. However, it is quite simple to solve this problem without any fancy footwork.

Solution 2.8

We saw in Section 2.3 that the payment per loss can be written as follows:

$$\begin{aligned} u &= \frac{L}{\alpha}+d = \frac{425}{0.85}+100=600 \\ Y &= \alpha\left((X-d)_+\right)-\alpha\left((X-u)_+\right)=0.85\left((X-100)_+-(X-600)_+\right) \\ &= \alpha\left(X\wedge u - X\wedge d\right)=0.85\left(X\wedge 600 - X\wedge 100\right) \end{aligned}$$

Solution 2.9

We have the general relation $E\left[Y^k\right] = E\left[Z^k\right]\Pr(Y > 0)$. As a result, we have:

$$\begin{aligned}
\text{var}(Y) &= E\left[Y^2\right] - \left(E[Y]\right)^2 = E\left[Z^2\right]\Pr(Y > 0) - \left(E[Z]\Pr(Y > 0)\right)^2 \\
&= E\left[Z^2\right]\Pr(Y > 0) - \left(E[Z]\right)^2 \Pr(Y > 0)^2 \\
&= E\left[Z^2\right]\Pr(Y > 0) - \left(E[Z]\right)^2 \left(\Pr(Y > 0) + \Pr(Y > 0)^2 - \Pr(Y > 0)\right) \\
&= \text{var}(Z)\Pr(Y > 0) + \left(E[Z]\right)^2 \Pr(Y > 0)\left(1 - \Pr(Y > 0)\right)
\end{aligned}$$

If I is an indicator for the event $Y > 0$, then the last line in the above equations is consistent with the part of the double expectation theorem that would say $\text{var}(Y) = E\left[\text{var}(Y|I)\right] + \text{var}\left(E\left[Y|I\right]\right)$.

Notice that $E\left[Y|I = 0\right] = \text{var}\left(Y|I = 0\right) = 0$ since you are given that $Y = 0$, and that $E\left[Y|I = 1\right] = E[Z]$, and $\text{var}\left(Y|I = 1\right) = \text{var}(Z)$ since $Z = Y$ when $Y > 0$. The point of this exercise is for you to notice that you **do not have** $\text{var}(Y)$ equal to $\text{var}(Z)\Pr(Y > 0)$.

Solution 2.10

The Pareto survival function is (see Section 2.2):

$$s_X(x) = \left(\frac{\theta}{x + \theta}\right)^{\alpha}$$

So from Theorem 2.3, we have:

$$\begin{aligned}
E[X \wedge d] &= \int_0^d s_X(x)\,dx = \int_0^d \left(\frac{\theta}{x + \theta}\right)^{\alpha} dx = \theta^{\alpha}\left(-\frac{1}{(x+\theta)^{\alpha-1}(\alpha-1)}\Bigg|_0^d\right) \\
&= \frac{\theta}{(\alpha-1)}\left(\frac{\theta^{\alpha-1}}{\theta^{\alpha-1}} - \frac{\theta^{\alpha-1}}{(d+\theta)^{\alpha-1}}\right) = \frac{\theta}{(\alpha-1)}\left(1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right)
\end{aligned}$$

Solution 2.11

Note first that we have:

$$Y = X \wedge 2{,}600 - X \wedge 100 \qquad \text{(Table 2.1)}$$

$$E\left[X \wedge d\right] = \theta\left(1 - e^{-d/\theta}\right) = 2{,}000\left(1 - e^{-d/2{,}000}\right) \qquad \text{(Table 2.3)}$$

From these formulas we have:

$$E[Y] = E[X \wedge 2600] - E[X \wedge 100] = 2{,}000\left(\left(1 - e^{-2{,}600/2{,}000}\right) - \left(1 - e^{-100/2{,}000}\right)\right) = 1{,}357.40$$

We also have $\Pr(Y > 0) = \Pr(X > 100) = e^{-100/2{,}000} = 0.95123$, so $E[Z] = E[Y]/0.95123 = 1{,}427.00$

Solution 2.12

From the given information, we have:

$$f(x) = \begin{cases} 0.40 f_1(x) & \text{if } 0 < x \le 1{,}000 \\ 0.60 f_2(x) & \text{if } 1{,}000 < x \end{cases}$$

where:

$$f_1(x) = 0.001 \quad \text{for } 0 < x \le 1{,}000 \quad \text{(uniform)}$$

$$f_2(x) = \frac{2 \times 1{,}000^2}{x^3} \quad \text{for } 1{,}000 < x \quad \text{(single parameter Pareto)}$$

We want to calculate:

$$E[Y] = E[X \wedge 2{,}000] = \int_0^{2{,}000} x\, f(x)\, dx + \int_{2{,}000}^{\infty} 2{,}000\, f(x)\, dx$$

$$= 0.4 \underbrace{\int_0^{1{,}000} 0.001 x\, dx}_{E[X_1] = 1{,}000/2} + 0.6 \underbrace{\left(\int_{1{,}000}^{2{,}000} \frac{2 \times 1{,}000^2\, x}{x^3} dx + 2{,}000 \Pr(X > 2{,}000) \right)}_{E[X_2 \wedge 2{,}000]}$$

$$= 200 + 0.6\left(-\frac{2 \times 1{,}000^2}{x} \Bigg|_{1{,}000}^{2{,}000} + 2{,}000 \left(\frac{1{,}000}{2{,}000}\right)^2 \right) = 1{,}100$$

Solution 2.13

Using the same data as in Solution 2.12 we need to calculate $E\left[Y^2\right]$:

$$E\left[Y^2\right] = E\left[(X \wedge 2{,}000)^2\right] = \int_0^{2{,}000} x^2 f(x)\,dx + \int_{2{,}000}^{\infty} 2{,}000^2 f(x)\,dx$$

$$= 0.4\underbrace{\int_0^{1{,}000} 0.001x^2\,dx}_{1{,}000^2/3} + 0.6\underbrace{\left(\int_{1{,}000}^{2{,}000} \frac{2\times 1{,}000^2\, x^2}{x^3}dx + 2{,}000^2 \Pr(X > 2{,}000)\right)}_{E\left[(X_2 \wedge 2{,}000)^2\right]}$$

$$= 133{,}333.33 + 0.6\left(2\times 1{,}000^2 \ln(x)\Big|_{1{,}000}^{2{,}000} + 2{,}000^2\left(\frac{1{,}000}{2{,}000}\right)^2\right)$$

$$= 1{,}565{,}110$$

Using the result in Solution 2.12, we have:

$$\text{var}(Y) = 1{,}565{,}110 - 1{,}100^2 = 355{,}110$$

For aggregate annual claims we have $S = Y_1 + \cdots + Y_{N_L}$:

$$E[S] = E[N_L]\,E[Y] = 80 \times 1{,}100 = 88{,}000$$

$$\text{var}(S) = E[N_L]\,\text{var}(Y) + (E[Y])^2\,\text{var}(N_L)$$

$$= 80 \times 355{,}110 + (1{,}100)^2 \times 120 = 173{,}608{,}800$$

Solution 2.14

The first thing we need to do is compute the gamma parameters:

$$E[X] = \alpha\theta = 100\ ,\quad \text{var}(X) = \alpha\theta^2 = 5{,}000 \quad\Rightarrow\quad \alpha = 2\ ,\ \theta = 50$$

With an **ordinary deductible** of $d = 50$ the payment per loss is $Y = (X - 50)_+$. Using Theorem 2.2 and Table 2.3, we have an expected payment per loss given by:

$$E[X \wedge 50] = \alpha\theta\,\Gamma(\alpha+1\,;\,d/\theta) + d\big(1 - \Gamma(\alpha\,;\,d/\theta)\big)$$

$$= 100\,\Gamma(3\,;\,50/50) + 50\big(1 - \Gamma(2\,;\,50/50)\big)$$

$$= 100\left(1 - e^{-1}\left(1+1+1^2/2!\right)\right) + 50\left(e^{-1}(1+1)\right) = 44.82$$

$$E[Y] = E[X] - E[X \wedge 50] = 100 - 44.82 = 55.18$$

To compute the expected payment per payment event we must divide $E[Y]$ by:

$$\Pr(Y > 0) = \Pr(X > 50) = 1 - \Gamma(\alpha = 2\,;\,50/\theta = 1) = 1 - \left(1 - e^{-1}(1+1)\right) = 0.73576$$

$$E[Z] = E[Y]/0.7356 = 75$$

With a **franchise deductible** of $d = 50$ the payment per loss is $Y = (X-50)_+ + 50 I_{50}(X)$ where $I_{50}(X)$ is an indicator for the event $X > 50$. So the expected payment per loss is:

$$E[Y] = E\left[(X-50)_+\right] + 50 \text{ Pr}(X > 50) = 91.97$$

The expected payment per payment event is thus:

$$E[Z] = \frac{E[Y]}{\Pr(Y>0)} = \frac{E\left[(X-50)_+\right] + 50 \text{ Pr}(X > 50)}{\Pr(X > 50)} = \frac{E\left[(X-50)_+\right]}{\Pr(X > 50)} + 50 = 75 + 50 = 125$$

Solution 2.15

You will need the expected limited loss formula for the Pareto family:

$$E[X \wedge d] = \frac{\theta}{\alpha - 1}\left(1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right) = \frac{500}{1}\left(1 - \left(\frac{500}{600}\right)^{1}\right) = \frac{500}{6}$$

$$\text{LER} = \frac{E[X \wedge 100]}{E[X]} = \frac{500/6}{500} = \frac{1}{6}$$

The easiest way to calculate the MEL $= E\left[X - 100 \mid X > 100\right]$ is to use the fact that $X - 100 \mid X > 100$ follows a 2-parameter Pareto distribution with $\alpha = 2$, $\theta^* = \theta + 100 = 600$. So we have:

$$\text{MEL} = E\left[X - 100 \mid X > 100\right] = \frac{\theta^*}{\alpha - 1} = 600$$

Solution 2.16

Using the results in Solution 2.15, the expected payment per loss this year is:

$$E\left[(X-100)_+\right] = E[X] - E[X \wedge 100] = 500 - \frac{500}{6} = 416.67$$

We are asked to calculate $E\left[(1.1X-100)_+\right]$ as the expected payment per loss next year:

Option 1. Use the fact that $1.1X$ is Pareto with $\alpha = 2$, $\theta^* = 1.1(500) = 550$. Therefore, we have:

$$E[1.1X \wedge 100] = \frac{\theta^*}{\alpha - 1}\left(1 - \left(\frac{\theta^*}{\theta^* + 100}\right)^{\alpha-1}\right) = 550\left(1 - \left(\frac{550}{650}\right)\right) = 84.62$$

$$E\left[(1.1X-100)_+\right] = E[X] - E[1.1X \wedge 100] = 550 - 84.62 = 465.38$$

$$\text{Percent Increase} = 100\left(\frac{465.38}{416.67} - 1\right) = 11.7\%$$

Option 2. Factor out the 1.1:

$$E\left[(1.1X-100)_+\right] = 1.1E\left[\left(X-\frac{100}{1.1}\right)_+\right] = 1.1\left(E[X]-E\left[X\wedge\frac{100}{1.1}\right]\right)$$
$$=1.1\left(\frac{500}{2-1}-\frac{500}{2-1}\left(1-\left(\frac{500}{500+(100/1.1)}\right)\right)\right)=465.38$$

Solution 2.17

The payment per loss is $Y=X\wedge 500$. The variance of Y can be computed with the help of the limited loss moments for the gamma distribution that are found in Tables 2.3 and 2.4:

$$E[X\wedge d]=\alpha\theta\Gamma(\alpha+1;\,d/\theta)\;+\;d\left(1-\Gamma(\alpha;\,d/\theta)\right)$$
$$E\left[(X\wedge d)^2\right]=\alpha(\alpha+1)\theta^2\,\Gamma(\alpha+2;\,d/\theta)\;+\;d^2\left(1-\Gamma(\alpha;\,d/\theta)\right)$$

Since $\alpha=2$, $\theta=250$, and $d=500$, we have:

$$\Gamma(\alpha;d/\theta)=\Gamma(2;2)=1-e^{-2}\left(1+\frac{2^1}{1!}\right)=0.59399$$
$$\Gamma(\alpha+1;d/\theta)=\Gamma(3;2)=1-e^{-2}\left(1+\frac{2^1}{1!}+\frac{2^2}{2!}\right)=0.32332$$
$$\Gamma(\alpha;d/\theta)=\Gamma(4;2)=1-e^{-2}\left(1+\frac{2^1}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}\right)=0.14288$$
$$E[X\wedge 500]=500\Gamma(3;\,2)\;+\;500\left(1-\Gamma(2;\,2)\right)=364.66$$
$$E\left[(X\wedge 500)^2\right]=2(3)250^2\,\Gamma(4;\,2)\;+\;500^2\left(1-\Gamma(2;\,2)\right)=155.080.16$$
$$\text{var}(Y)=22{,}100$$

Solution 2.18

The payment per loss is $Y=(X-50)_+-(X-550)_+$ since $u=d+L/\alpha=50+500/1=550$. For an exponential distribution we have:

$$E\left[(X-d)_+\right]=E[X]-E[X\wedge d]=\theta-\theta\left(1-e^{-d/\theta}\right)=\theta\;e^{-d/\theta}$$
$$E[Y]=E\left[(X-50)_+\right]-E\left[(X-550)_+\right]=500\left(e^{-0.1}-e^{-1.1}\right)=285.98$$

From part (iv) of Theorem 2.4, we have:

$$E\left[Y^2\right] = E\left[\left((X-50)_+\right)^2\right] - E\left[\left((X-550)_+\right)^2\right] - 2(550-50)E\left[(X-550)_+\right]$$

$$= \underbrace{E\left[(X-50)^2 \mid X>50\right]}_{\substack{2\left(500^2\right)\text{ conditional} \\ \text{exponential distribution}}} \underbrace{\Pr(X>50)}_{e^{-50/500}} - \underbrace{E\left[(X-550)^2 \mid X>550\right]}_{\substack{2\left(500^2\right)\text{ conditional} \\ \text{exponential distribution}}} \underbrace{\Pr(X>550)}_{e^{-550/500}}$$

$$-1{,}000 \underbrace{E\left[X-550 \mid X>550\right]}_{\substack{500 \quad \text{conditional} \\ \text{exponential distribution}}} \underbrace{\Pr(X>550)}_{e^{-550/500}}$$

$$= 119{,}547.63$$

We now have $\text{var}(Y) = 37{,}761.25$. Aggregate annual claims are $S = Y_1 + \cdots + Y_{N_L}$ where $E[N_L] = 50$ and $\text{var}(N_L) = 100$. From compound sum moment formulas we have:

$$E[S] = E[N_L]\, E[Y] = 50 \times 285.98 = 14{,}299.16$$

$$\text{var}(S) = E[N_L]\, \text{var}(Y) + \left(E[Y]\right)^2 \, \text{var}(N_L)$$

$$= 50 \times 37{,}761.25 + (285.98)^2 \times 100 = 10{,}066{,}700$$

Solution 2.19

Using results from Solution 2.18, we have:

$$\Pr\left(S > 1.25E[S]\right) = \Pr\left(\frac{S-E[S]}{\sqrt{\text{var}(S)}} > \frac{1.25E[S]-E[S]}{\sqrt{\text{var}(S)}}\right) \approx 1 - \Phi\left(\frac{0.25E[S]}{\sqrt{\text{var}(S)}}\right)$$

$$= 1 - \Phi(1.13) \approx 1 - \left(0.7\,\Phi(1.1) + 0.3\,\Phi(1.2)\right) = 0.130$$

Solution 2.20

We saw in Solution 2.14 that $\alpha = 2$ and $\theta = 50$. We also calculated $E\left[(X-50)_+\right] = 55.18$. Here is another way to duplicate the expected value calculation and to speed up the second moment calculation. We will identify the distribution of $Z = X - 50 \mid X > 50$ as a 50/50 mixture of an exponential with $\theta = 50$ and a gamma with $\alpha = 2$ and $\theta = 50$:

$$Z = X - 50 \mid X > 50$$

$$s_Z(z) = {}_zp_{50} = \frac{s_X(50+z)}{s_X(50)} = \frac{1 - \Gamma\left(2;(50+z)/50\right)}{1 - \Gamma\left(2;50/50\right)} = \frac{e^{-(50+z)/50}\left(1 + \dfrac{\left((50+z)/50\right)^1}{1!}\right)}{e^{-50/50}\left(1 + \dfrac{1^1}{1!}\right)}$$

$$= e^{-z/50}\left(\frac{2 + z/50}{2}\right) = 0.50\,e^{-z/50} + 0.50\,e^{-z/50}\left(1 + z/50\right)$$

This final expression is a weighted average of an exponential survival function and a gamma survival function. So moments about the origin of Z can be computed as weighted averages of exponential and gamma moments:

$$E[Z] = 0.50(50) + 0.50(100) = 75$$
$$E\left[Z^2\right] = 0.50\left(2 \times 50^2\right) + 0.50\left(2 \times 3 \times 50^2\right) = 10{,}000$$

Now multiply by $\Pr(Y>0) = \Pr(X>50) = 1 - \Gamma(2;50/50) = 0.73576$ to obtain moments of Y:

$$E[Y] = 0.73576 \times E[Z] = 55.18 \quad , \quad E\left[Y^2\right] = 0.75376 \times E\left[Z^2\right] = 7{,}357.59$$
$$\text{var}(Y) = 4{,}312.54$$

