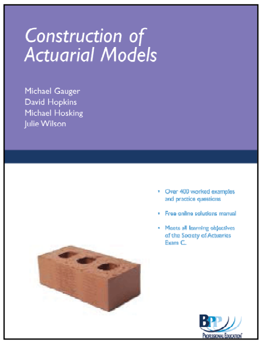

Construction of Actuarial Models

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Solutions to practice questions – Chapter 3

Solution 3.1

The interval $\left(-\infty, \tilde{\theta}+1.645\sigma_{\tilde{\theta}}\right)$ is a 95% confidence interval for θ since:

$$\begin{aligned}
\Pr\left(-\infty<\theta<\tilde{\theta}+1.645\sigma_{\tilde{\theta}}\right) &= \Pr\left(-\tilde{\theta}-1.645\sigma_{\tilde{\theta}}<-\theta<\infty\right) \\
&= \Pr\left(-1.645\sigma_{\tilde{\theta}}<\tilde{\theta}-\theta<\infty\right) \\
&= \Pr\left(-1.645<\frac{\tilde{\theta}-\theta}{\sigma_{\tilde{\theta}}}<\infty\right) \\
&= \Pr\left(N(0,1)>-1.645\right) \\
&= 0.95
\end{aligned}$$

Solution 3.2

To obtain an equal-tailed 99% confidence interval for μ, we just replace 1.960 by 2.576, the upper 0.5% point of $N(0,1)$. This gives the interval:

$$\left(12.500-2.576\times\frac{6}{\sqrt{25}}, 12.500+2.576\times\frac{6}{\sqrt{25}}\right)=(9.409,\ 15.591)$$

The width of this interval is:

$$2\times 2.576\times\frac{6}{\sqrt{25}}=6.182$$

Note that we can increase the level of confidence only by increasing the width of the confidence interval.

Solution 3.3

The width of such a 95% confidence interval is $2\times 1.960\times\dfrac{12}{\sqrt{n}}$. If the width is to be at most 10, then we require:

$$2\times 1.960\times\frac{12}{\sqrt{n}}\le 10$$

We can rearrange this to obtain:

$$\sqrt{n}\ge 4.704\,, \qquad ie \quad n\ge 22.128$$

So we would require a sample of at least 23 observations to obtain an equal sided 95% confidence interval of width at most 10.

Solution 3.4

We require:

$$2\times 2.576\times\frac{12}{\sqrt{n}}\le 10$$

So:

$$\sqrt{n}\ge\frac{2\times 2.576\times 12}{10}=6.182\Rightarrow n\ge 38.22$$

So the minimum sample size is 39.

Solution 3.5

An equal -tailed 95% confidence interval for μ is:

$$\left(45.02-1.960\times\frac{10.13}{\sqrt{40}},\ 45.02+1.960\times\frac{10.13}{\sqrt{40}}\right)=(41.88,\ 48.16)$$

Solution 3.6

The upper 10% point of the standard normal distribution is 1.282. So the upper limit of the confidence interval is:

$$b=\bar{x}+1.282\frac{\sigma}{\sqrt{n}}=45.02+1.282\times\frac{10.13}{\sqrt{40}}=47.07$$

Solution 3.7

We have $n=8$ and $s^2=386.125$.

The upper and lower 2.5% points of χ_7^2 are 1.690 and 16.01, respectively.

So an equal-tailed 95% confidence interval for the population variance is:

$$\left(\frac{(n-1)s^2}{\chi^2_{0.975,n-1}},\ \frac{(n-1)s^2}{\chi^2_{0.025,n-1}}\right)=\left(\frac{7\times 386.125}{16.01},\ \frac{7\times 386.125}{1.690}\right)$$

$$=(168.82,\ 1599.33)$$

Solution 3.8

The width of the confidence interval is:

$$1,599.33 - 168.82 = 1,430.51$$

Solution 3.9

The sample mean is $\bar{x} = \dfrac{5,125}{50} = 102.5$ and the sample variance is $s^2 = \dfrac{1}{49}\left(526,342.5 - 50 \times 102.5^2\right) = 21.02041$.

An equal-tailed 99% confidence interval for the population mean is:

$$102.5 \pm 2.576\sqrt{\frac{21.02041}{50}} = 102.5 \pm 1.67 = (100.83, 104.17)$$

Solution 3.10

An equal-tailed 95% confidence interval for the population variance is:

$$\left(\frac{49 \times 21.02041}{70.222}, \frac{49 \times 21.02041}{31.555}\right) = (14.668, 32.641)$$

Solution 3.11

We are told that $\sum_{i=1}^{n}(x_i - \bar{x})^2 = (n-1)s^2 = 2,016$. So:

$$\frac{2,016}{\chi^2_{0.025,n-1}} = 8.7855^2 \Rightarrow \chi^2_{0.025,n-1} = 26.119$$

and: $$\frac{2,016}{\chi^2_{0.975,n-1}} = 18.9247^2 \Rightarrow \chi^2_{0.975,n-1} = 5.629$$

From the *Tables* we see that $n - 1 = 14$. So $n = 15$.

Solution 3.12

The likelihood function is:

$$L = q$$

Multiplying this by the prior pdf, we have:

$$f_{post}(q) \propto q^3 \qquad 0 < q < 1$$

So the posterior distribution of q is beta with parameters $a = 4$ and $b = 1$.

Integrating the posterior pdf between 0 and h gives:

$$\int_0^h \frac{\Gamma(5)}{\Gamma(4)\Gamma(1)} x^3\, dx = \int_0^h 4x^3\, dx = h^4$$

Setting this expression equal to 0.025 gives $h = 0.3976$, and setting it equal to 0.975 gives $h = 0.9937$.

So $(0.3976, 0.9937)$ is a 95% Bayesian confidence interval for q.

Solution 3.13

The likelihood function is now:

$$L = q\left(1-q\right)$$

Multiplying this by the prior pdf, we have:

$$f_{post}(q) \propto q^3(1-q) \qquad 0 < q < 1$$

So the posterior distribution of q is beta with parameters $a = 4$ and $b = 2$.

Integrating the posterior pdf between 0 and h gives:

$$\int_0^h \frac{\Gamma(6)}{\Gamma(4)\Gamma(2)} x^3(1-x)\,dx = \int_0^h \left(20x^3 - 20x^4\right)dx = 5h^4 - 4h^5$$

When $h = 0.2836$, $5h^4 - 4h^5 = 0.0250$. Also, when $h = 0.9473$, $5h^4 - 4h^5 = 0.9750$. So $\left(0.2836, 0.9473\right)$ is a 95% Bayesian confidence interval for q.

Solution 3.14

The likelihood function is:

$$L = \prod_{i=1}^{100} f(x_i) = \prod_{i=1}^{100} \frac{1}{\theta} e^{-x_i/\theta} = \frac{1}{\theta^{100}} e^{-\sum_{i=1}^{100} x_i \big/ \theta} = \frac{1}{\theta^{100}} e^{-100\bar{x}/\theta}$$

Taking logs:

$$\ln L = -100\ln\theta - \frac{100\bar{x}}{\theta}$$

Differentiating with respect to θ:

$$\frac{d\ln L}{d\theta} = -\frac{100}{\theta} + \frac{100\bar{x}}{\theta^2}$$

Setting this equal to 0:

$$\frac{100}{\theta} = \frac{100\bar{x}}{\theta^2} \Rightarrow \theta = \bar{x} = 800$$

The second derivative of the log-likelihood is:

$$\frac{d^2\ln L}{d\theta^2} = \frac{100}{\theta^2} - \frac{200\bar{x}}{\theta^3} = -0.00015625 \text{ when } \theta = 800$$

So the maximum likelihood estimate of θ is $\hat{\theta} = 800$. The corresponding maximum likelihood estimator of θ is $\tilde{\theta} = \bar{X}$. This estimator is asymptotically normally distributed and its variance is estimated by:

$$\frac{1}{0.00015626} = 6400$$

So a 95% confidence interval for θ is:

$$800 \pm 1.960\sqrt{6400} = 800 \pm 156.8 = \left(643.2, 956.8\right)$$

Solution 3.15

The pdf is:

$$f(x)=\frac{\alpha(500)^{\alpha}}{x^{\alpha+1}}$$

So the likelihood function is:

$$L=\prod_{i=1}^{100}\frac{\alpha\left(500\right)^{\alpha}}{x_i^{\alpha+1}}=\frac{\alpha^{100}\,500^{100\alpha}}{\prod x_i^{\alpha+1}}$$

Taking logs:

$$\ln L=100\ln\alpha+100\alpha\ln 500-\left(\alpha+1\right)\sum_{i=1}^{100}\ln x_i$$

Differentiating with respect to α:

$$\frac{d\ln L}{d\alpha}=\frac{100}{\alpha}+100\ln 500-\sum_{i=1}^{100}\ln x_i$$

Setting this equal to 0:

$$\frac{100}{\alpha}+100\ln 500-\sum_{i=1}^{100}\ln x_i=0\Rightarrow\alpha=\frac{100}{\sum\ln x_i-100\ln 500}$$

The second derivative is:

$$\frac{d^2\ln L}{d\alpha^2}=-\frac{100}{\alpha^2}$$

So the maximum likelihood estimate of α is:

$$\hat{\alpha}=\frac{100}{658.617-100\ln 500}=2.691$$

and the estimated standard error is:

$$\sqrt{\frac{\hat{\alpha}^2}{100}}=0.269$$

So a 99% confidence interval for α is:

$$2.691\pm 2.576\times 0.269=\left(1.998,3.384\right)$$

Solution 3.16

The sample mean of all the data is:

$$\frac{48\times 10.78+22.05}{49}=11.01$$

Also:

$$\frac{1}{47}\left(\sum_{i=1}^{48}x_i^2-48\times 10.78^2\right)=480.56\Rightarrow \sum_{i=1}^{48}x_i^2=28,164.3232$$

Adding in the 49th observation:

$$\sum_{i=1}^{49}x_i^2=28,650.5257$$

So:

$$s^2=\frac{1}{48}\left(28,650.5257-49\times 11.01^2\right)=473.1404$$

A 90% confidence interval for μ is then:

$$11.01\pm 1.645\sqrt{\frac{473.1404}{49}}=11.01\pm 5.11=(5.90,16.12)$$

Solution 3.17

The posterior distribution for λ is given by the conditional probabilities $\Pr(\lambda=j\,|\,N=4)$ for $j=1,2,3,4,5$. Now:

$$\Pr(\lambda=j\,|\,N=4)=\frac{\Pr(\lambda=j,N=4)}{\Pr(N=4)}=\frac{\Pr(N=4\,|\,\lambda=j)\Pr(\lambda=j)}{\Pr(N=4)}$$

When $j=1$, the numerator is:

$$\Pr(N=4\,|\,\lambda=1)\Pr(\lambda=1)=\frac{e^{-1}1^4}{4!}\times 0.1=0.00153$$

Similarly:

$$\Pr(N=4\,|\,\lambda=2)\Pr(\lambda=2)=0.018045$$
$$\Pr(N=4\,|\,\lambda=3)\Pr(\lambda=3)=0.050409$$
$$\Pr(N=4\,|\,\lambda=4)\Pr(\lambda=4)=0.058610$$
$$\Pr(N=4\,|\,\lambda=5)\Pr(\lambda=5)=0.017547$$

The denominator is:

$$\Pr(N=4)=\sum_{j=1}^{5}\Pr(N=4\,|\,\lambda=j)\Pr(\lambda=j)=0.146144$$

So the posterior distribution for λ is:

λ	1	2	3	4	5
posterior probability	0.0105	0.1235	0.3449	0.4010	0.1201

The narrowest 95% Bayesian confidence interval for λ is [2,5].

Solution 3.18

To find the upper limit of this confidence interval, we require the lower 5% of the χ_8^2 distribution. From the *Tables*, this is 2.733. So the upper limit is:

$$b = \frac{8 \times 57}{2.733} = 166.850$$

Solution 3.19

The upper 5% point of χ_8^2 is 15.507. So a 95% confidence interval for the variance of the population is:

$$\left(\frac{8 \times 57}{15.507}, \infty \right) = (29.406, \infty)$$

Hence, a 95% confidence interval for the standard deviation of the population is:

$$\left(\sqrt{29.406}, \infty \right) = (5.423, \infty)$$

Solution 3.20

The width of the confidence interval is:

$$2 \times 1.960 \times \frac{10.204}{\sqrt{n}} = 5$$

So:

$$\sqrt{n} = \frac{2 \times 1.960 \times 10.204}{5} = 8 \Rightarrow n = 64$$