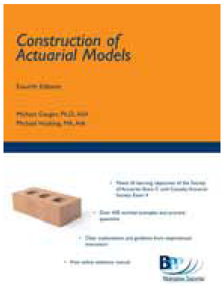

Construction of Actuarial Models

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Solutions to practice questions – Chapter 3

Solution 3.1

From the given information we have:

$$\frac{6/64}{1/16} = \frac{\Pr(N=1)}{\Pr(N=0)} = a + \frac{b}{1} \quad , \quad \frac{27/256}{6/64} = \frac{\Pr(N=2)}{\Pr(N=1)} = a + \frac{b}{2}$$

The solution of this simultaneous system of equations is: $a = b = 0.75$.

Solution 3.2

The value of a is positive only for a negative binomial distribution. (It is zero for a Poisson distribution, and negative for a binomial distribution.) Now use the form of a and b for a negative binomial to determine the parameters:

$$0.75 = a = \frac{\beta}{1+\beta} \quad , \quad 0.75 = b = \frac{(r-1)\beta}{1+\beta} \quad \Rightarrow \quad r = 2 \ , \ \beta = 3$$

Now it is easy to finish the exercise:

$$E[N] = r\beta = 6 \quad , \quad \operatorname{var}(N) = r\beta(1+\beta) = 24$$

$$\Pr(N=8) = \frac{r(r+1)\cdots(r+7)}{8!}\left(\frac{\beta}{1+\beta}\right)^8 \left(\frac{1}{1+\beta}\right)^2 = \frac{9 \times 3^8}{4^{10}}$$

Solution 3.3

The Poisson distribution with $\lambda = 1.4$ would be a good model for N, the random number of accidents per month. With this assumption the probability of more than 1 accident in a month is:

$$\Pr(N \geq 2) = 1 - \Pr(N=0) - \Pr(N=1) = 1 - e^{-1.4}(1+1.4) = 0.40817$$

Now let M be the number of months in the next 6 months with more than 1 accident.

A good model for the distribution of M is binomial with $m=6$ trials (each month is a trial). A month is considered to be a "success" if more than 1 accident occurs. The probability of "success" is:

$$q = \Pr(\text{"success"}) = \Pr(N \geq 2) = 0.40817$$

The probability of more than 1 success in the next 6 trials is:

$$\begin{aligned}\Pr(M \geq 2) &= 1 - \Pr(M=0) - \Pr(M=1) \\ &= 1 - \binom{6}{0}(0.40817)^0 (0.59183)^6 - \binom{6}{1}(0.40817)^1 (0.59183)^5 \\ &= 0.77921\end{aligned}$$

Solution 3.4

The probability that a loss exceeds 900 is: $\Pr(X > 900) = \int_{900}^{1,000} 0.001\, dx = 0.10$. Consider a loss to be a success if it exceeds 900. Due to the assumptions in the question, the losses can be viewed as being a series of independent Bernoulli trials with $q = \Pr(X > 900) = 0.10$. Let N be the number of failures observed before 2 successes are observed. Then the number of losses observed is $N+2$, and N follows a negative binomial distribution with parameters $r=2$ and $\beta = 0.10^{-1} - 1 = 9$). So the expected number of losses observed is:

$$E[2+N] = 2 + r\beta = 2 + 2\times 9 = 20$$

Solution 3.5

The given density function is for a gamma distribution with $\alpha = 3$, $\theta = 1/5$. We are given that $N \mid \Lambda = \lambda$ is Poisson with mean $\Lambda = \lambda$, and that Λ follows a gamma distribution. As a result, we know that N, the annual number of accidents on a randomly selected 10-mile stretch of this highway, follows a negative binomial distribution with $r=\alpha=3$ and $\beta=\theta=1/5$. The annual number of accidents on a 20-mile stretch of highway, M, can be viewed as a sum of 2 such negative binomial distribution. So it follows a negative binomial distribution with $r = 2\times 3 = 6$ and $\beta = \theta = 1/5$. So the probability of $M=2$ is:

$$\Pr(M=2) = \frac{6\times 7}{2!}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^6 = 0.19536$$

Solution 3.6

The number of consecutive road games that they will lose follows a geometric distribution with $p = 1/8$ $(\beta = 7)$. We are asked to compute $E[N \mid N \geq 6]$. According to the memory-less property, $N-6 \mid N \geq 6$ also follows this same geometric distribution. So we have:

$$\begin{aligned}E[N \mid N \geq 6] &= E[6 + N - 6 \mid N \geq 6] = 6 + E[N-6 \mid N \geq 6] = 6 + E[N] \\ &= 6 + \beta = 6 + \left(p^{-1} - 1\right) = 6 + 8 - 1 = 13\end{aligned}$$

Solution 3.7

Each of the $m = 25$ lives is viewed as a Bernoulli trial. A trial is considered to be a success if the policyholder dies within 5 years. The probability of success is:

$$q = {}_5q_{50} = 1 - \frac{l_{55}}{l_{50}} = 1 - \frac{90-55}{90-50} = \frac{5}{40} \quad \text{since } l_x = 90 - x$$

The number of deaths from this group in the next 5 years follows a binomial distribution with $m = 25$, $q = 1/8$. We are asked to determine:

$$\Pr(M \geq 2) = 1 - \Pr(M=0) - \Pr(M=1) = 1 - \binom{25}{0}\left(\frac{1}{8}\right)^0\left(\frac{7}{8}\right)^{25} - \binom{25}{0}\left(\frac{1}{8}\right)^0\left(\frac{7}{8}\right)^{25}$$

$$= 1 - 0.03550 - 0.12678 = 0.83772$$

Solution 3.8

We have a compound counting model for the annual number of payments to dependents:

$$C = M_1 + \cdots + M_N \quad \text{where } N \sim \text{negative binomial } r = \beta = 2$$

From the given information it is easily checked that $E[M] = 1$, $E\left[M^2\right] = 1.6$, $\text{var}(M) = 0.6$. So from standard compound sum moment formulas, we have:

$$E[C] = E[N]\,E[M] = (r\beta)1 = 4$$

$$\text{var}(C) = E[N]\,\text{var}(M) + \left(E[M]\right)^2 \text{var}(N)$$

$$= (r\beta) \times 0.6 \; + (1.0)^2 \left(r\beta(1+\beta)\right) = 14.4$$

Solution 3.9

We need to use Theorem 3.1:

$$r = 2,\ \beta = 2 \;\Rightarrow\; a = \frac{\beta}{1+\beta} = \frac{2}{3}\ ,\ b = \frac{(r-1)\beta}{1+\beta} = \frac{2}{3}$$

and $P_N(z) = \left(1 - \beta(z-1)\right)^{-r} = (3-2z)^{-2}$

The starting value for the recursion is:

$$\Pr(C=0) = P_N\left(\Pr(M=0)\right) = P_N(0.3) = \left(3 - 2(0.3)\right)^{-2} = 0.17361$$

The recursion formula is:

$$\begin{aligned}\Pr(C=n) &= \frac{1}{1-a\Pr(M=0)}\sum_{j=1}^{n}\left(a+\frac{bj}{n}\right)\Pr(M=j)\Pr(C=n-j)\\ &= \frac{1}{1-\frac{2}{3}(0.3)}\sum_{j=1}^{n}\left(\frac{2}{3}+\frac{2j}{3n}\right)\Pr(M=j)\Pr(C=n-j)\\ &= 1.25\left(\left(\frac{2}{3}+\frac{2}{3n}\right)0.4\Pr(C=n-1)+\left(\frac{2}{3}+\frac{4}{3n}\right)0.3\Pr(C=n-2)\right)\end{aligned}$$

With $n=1$, we have:

$$\Pr(C=1) = 1.25\left(\left(\frac{2}{3}+\frac{2}{3}\right)0.4\Pr(C=0)\right) = 0.11574$$

Finally, we have $\Pr(C \le 1) = \Pr(C=0) + \Pr(C=1) = 0.28935$

Solution 3.10

First let $\tilde{M} = M + 1$. This variable will also count the initial earthquake in an event. The annual number of earthquakes and aftershocks is:

$$C = \tilde{M}_1 + \cdots + \tilde{M}_N \quad \text{where } N \sim \text{geometric with } 2 = \beta = E[N]$$

The probability distribution of $\tilde{M}$ is:

$$\Pr(\tilde{M}=1) = 0.10 \;,\; \Pr(\tilde{M}=2) = 0.60 \;,\; \Pr(\tilde{M}=3) = 0.30$$

The moment generating function of the geometric primary distribution is:

$$P_N(z) = (1-\beta(z-1))^{-1} = (3-2z)^{-1}$$

The probability of C equal to zero is:

$$\Pr(C=0) = P_N\left(\Pr(\tilde{M}=0)\right) = P_N(0) = 3^{-1} = 1/3$$

We can use the recursion formula to calculate $\Pr(C=1)$. The compound geometric recursion formula is:

$$\begin{aligned}\Pr(C=n) &= \frac{\beta}{1+\beta\Pr(\tilde{M}\ge 1)}\sum_{j=1}^{n}\Pr(\tilde{M}=j)\Pr(C=n-j)\\ &= \frac{2}{1+2\times 1}\left(0.10\Pr(C=n-1)+0.60\Pr(C=n-2)+0.30\Pr(C=n-3)\right)\end{aligned}$$

So we have:

$$\Pr(C=1) = \frac{2}{3}\left(0.10\Pr(C=0)\right) = 0.02222$$

As a result, we have:

$$\Pr(C \geq 2) = 1 - \Pr(C = 0, 1) = 1 - 0.33333 - 0.02222 = 0.64444$$

Solution 3.11

Since $N \mid \Lambda$ is Poisson with mean Λ, we know that $E[N \mid \Lambda] = \text{var}(N \mid \Lambda) = \Lambda$. Applying the double expectation theorem, we have:

$$E[N] = E\left[E[N \mid \Lambda]\right] = E[\Lambda] = 2$$
$$\text{var}(N) = E\left[\text{var}(N \mid \Lambda)\right] + \text{var}\left(E[N \mid \Lambda]\right) = E[\Lambda] + \text{var}(\Lambda) = 2 + 2 = 4$$

Solution 3.12

We are given:

$$\Pr(N \mid \Lambda = \lambda) = e^{-\lambda}\,\frac{\lambda^n}{n!} \quad \text{for } n = 0, 1, 2, \ldots$$
$$\Pr(\Lambda = \lambda) = e^{-2}\,\frac{2^\lambda}{\lambda!} \quad \text{for } \lambda = 0, 1, 2, \ldots$$

So the probability that $N = 0$ is:

$$\Pr(N = 0) = \sum_{\lambda=0}^{\infty} \Pr(N=0 \mid \Lambda = \lambda)\,\Pr(\Lambda = \lambda)$$
$$= \sum_{\lambda=0}^{\infty} e^{-\lambda}\, e^{-2}\,\frac{2^\lambda}{\lambda!} = e^{-2}\sum_{\lambda=0}^{\infty}\frac{\left(2e^{-1}\right)^\lambda}{\lambda!} = e^{-2}\,e^{2e^{-1}} = 0.28245$$

Solution 3.13

For year 2008 we have ${}_{08}N_L$ is distributed as negative binomial where:

$$6 = E[{}_{08}N_L] = r_{08}\,\beta_{08} \;,\; 24 = \text{var}({}_{08}N_L) = r_{08}\,\beta_{08}\,(1+\beta_{08})$$
$$\Rightarrow\; r_{08} = 2 \;,\; \beta_{08} = 3$$

From the results of Section 3.5, a 10% increase in exposure will result in ${}_{09}N_L$ following a negative binomial distribution with $r_{09} = 1.10 r_{08} = 2.2$, $\beta_{09} = \beta_{08} = 3$.

From results in Chapter 2 we know that $X_{09} = 1.04X_{08}$ will follow a 2-parameter Pareto distribution with parameters $\alpha_{09} = \alpha_{08} = 2$ and $\theta_{09} = 1.04\theta_{08} = 1.04 \times 500 = 520$. With an ordinary deductible of 100 per loss in 2009, the probability that a loss event is a payment event is:

$$\upsilon = \Pr(X_{09} > 100) = s_{X_{09}}(100) = \left(\frac{520}{520 + 100}\right)^2 = 0.70343$$

From results in Section 3.6 we know that the distribution of *claim payments* in 2009, ${}_{09}N_P$, is negative binomial with:

$$r = r_{09} = 2.2\ ,\ \ {}_P\beta_{09} = 3\upsilon = 2.11030$$

Solution 3.14

Since losses are fully reimbursed in Year 2008, the expected annual claims payments are:

$$E[{}_{08}N_L]\,E[X_{08}] = 6 \times 500 = 3{,}000$$

The expected annual claims payments in Year 2009 can be computed in 2 different ways:

- $$E[{}_{09}N_L]\,E\left[(X_{09} - 100)_+\right] = (2.2 \times 3)\left(E[X_{09}] - E[X_{09} \wedge 100]\right)$$
$$= 6.6 \times \underbrace{\left(520 - 520\left(1 - \left(\frac{520}{520 + 100}\right)^{2-1}\right)\right)}_{\text{Pareto: } \frac{\theta_{09}}{\alpha_{09}-1}\left(1 - \left(\frac{\theta_{09}}{\theta_{09}+d}\right)^{\alpha_{09}-1}\right)} = 2{,}878.45$$

- $$E[{}_{09}N_P]\,E\big[\underbrace{X_{09} - 100 \mid X > 100}_{\text{Pareto: } \alpha = 2\ ,\ \theta = 620}\big] = (2.2 \times 2.11030)620 = 2{,}878.45$$

The percent change is –4.052%.

Solution 3.15

The expected annual claims payments in 2008 are 3,000. The expected annual claims payments in year 2009 with an ordinary deductible of *d* per loss are:

$$E[{}_{09}N_L]\,E[(1.04X_{08} - d)_+] = E[{}_{09}N_L]\left(E[1.04X_{08}] - E\left[\underbrace{(1.04X_{08})}_{\substack{\text{Pareto: } \alpha = 2 \\ \text{and } \theta = 520}} \wedge d\right]\right)$$

$$= (2.2 \times 3)\left(1.04 \times 500 - \frac{520}{2-1}\left(1 - \left(\frac{520}{520 + d}\right)^{2-1}\right)\right)$$

Setting this expression equal to 3,000 results in $d = 74.88$.

Solution 3.16

In the original form we have:

$$C = M_1 + \cdots + M_N \quad \text{where } N \sim \text{negative binomial } r = \beta = 2$$
$$\text{and: } \Pr(M=0) = 0.30\text{ , }\Pr(M=1) = 0.40\text{ , }\Pr(M=2) = 0.30$$

Now let $\tilde{M} = M \mid M > 0$: $\Pr(\tilde{M}=1) = (0.40/0.70)$, $\Pr(\tilde{M}=2) = (0.30/0.70)$. According to the results found in Section 3.6, the frequency of non-zero terms in the original compound sum, $\tilde{N}$, is negative binomial with parameters:

$$r = 2\text{ , }\beta^* = \beta\upsilon = 2\Pr(M > 0) = 2 \times 0.70 = 1.4$$

In the zero-filtered form we have:

$$C = \tilde{M}_1 + \cdots + \tilde{M}_{\tilde{N}}$$

Solution 3.17

From the results in Solution 3.16, we have:

$$E[C] = E[\tilde{N}]\, E[\tilde{M}] = (r\beta^*)\left(1 \times \frac{4}{7} + 2 \times \frac{3}{7}\right) = 4.0$$

$$\text{var}(C) = \underbrace{E[\tilde{N}]}_{r\beta^* = 2.8}\ \underbrace{\text{var}(\tilde{M})}_{\frac{16}{7} - \left(\frac{10}{7}\right)^2} + \underbrace{\left(E[\tilde{M}]\right)^2}_{\left(\frac{10}{7}\right)^2}\ \underbrace{\text{var}(\tilde{N})}_{r\beta^*(1+\beta^*) = 6.72} = 14.40$$

Solution 3.18

From the results in Solution 3.16, we have:

$$\Pr(C=0) = \Pr(\tilde{N}=0) = \left(\frac{1}{1+\beta^*}\right)^r = \left(\frac{1}{2.4}\right)^2 = 0.17361$$

$$\Pr(C=1) = \Pr(\tilde{N}=1)\Pr(\tilde{M}=1) = \left(\frac{2}{1}\left(\frac{\beta^*}{1+\beta^*}\right)^1\left(\frac{1}{1+\beta^*}\right)^r\right) \times \frac{4}{7} = 0.11574$$

Solution 3.19

Annual claims payments are $S = X_1 \wedge 250 + \cdots + X_{N_L} \wedge 250$. It is easy to compute the moments of the payment per loss variable using the formulas in Tables 2.3 and 2.4:

$$E[X \wedge 250] = \theta\left(1 - e^{-250/\theta}\right) = 100\left(1 - e^{-2.5}\right) = 91.79 \quad \text{(See Table 2.3)}$$

$$E\left[(X \wedge 250)^2\right] = 2\theta^2\,\Gamma(3;250/\theta) + 250^2\left(e^{-250/\theta}\right)$$

$$= 20{,}000\left(1 - e^{-2.5}\left(1 + 2.5 + \frac{2.5^2}{2!}\right)\right) + 250^2\,e^{-2.5} = 14{,}254.05010$$

Since the frequency model is Poisson with mean (and variance equal to 20), we have:

$$E[S] = 20\,E[X \wedge 250] = 1{,}835.83$$

$$\text{var}(S) = 20\,E\left[(X \wedge 250)^2\right] = 285{,}081$$

Solution 3.20

The first step is to determine d such that:

$$E\left[(X-d)_+\right] = E[X \wedge 250] = 91.79150$$

Form an exponential distribution formula in Table 2.3, we have:

$$91.79150 = \text{E}\left[(X-d)_+\right] = \theta\,e^{-d/\theta} = 100\,e^{-d/100} \;\Rightarrow\; d = 8.56505$$

Since the frequency of losses is Poisson, the variance in annual claims payments is:

$$\text{var}(S) = 20\,E\left[(X-d)_+^2\right] = 20 \times \underbrace{E\left[(X-d)^2 \mid X > d\right]}_{\substack{\text{conditional exponential} \\ \text{second moment is } 2(100)^2}} \; \underbrace{\Pr(X > d)}_{\substack{e^{-d/100} \\ = 0.91792}}$$

$$= 367{,}166$$

