



Construction of Actuarial Models

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by Mike Gauger and Michael Hosking

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Solutions to practice questions – Chapter 3

Solution 3.1

From the given information we have:

$$\frac{6/64}{1/16} = \frac{\Pr(N=1)}{\Pr(N=0)} = a + \frac{b}{1} \quad , \quad \frac{27/256}{6/64} = \frac{\Pr(N=2)}{\Pr(N=1)} = a + \frac{b}{2}$$

The solution of this simultaneous system of equations is: $a = b = 0.75$.

Solution 3.2

The value of a is positive only for a negative binomial distribution. (It is zero for a Poisson distribution, and negative for a binomial distribution.) Now use the form of a and b for a negative binomial to determine the parameters:

$$0.75 = a = \frac{\beta}{1 + \beta} \quad , \quad 0.75 = b = \frac{(r-1)\beta}{1 + \beta} \quad \Rightarrow \quad r = 2 \quad , \quad \beta = 3$$

Now it is easy to finish the exercise:

$$E[N] = r\beta = 6 \quad , \quad \text{var}(N) = r\beta(1 + \beta) = 24$$

$$\Pr(N=8) = \frac{r(r+1)\cdots(r+7)}{8!} \left(\frac{\beta}{1 + \beta}\right)^8 \left(\frac{1}{1 + \beta}\right)^2 = \frac{9 \times 3^8}{4^{10}}$$

Solution 3.3

The Poisson distribution with $\lambda = 1.4$ would be a good model for N , the random number of accidents per month. With this assumption the probability of more than 1 accident in a month is:

$$\Pr(N \geq 2) = 1 - \Pr(N=0) - \Pr(N=1) = 1 - e^{-1.4}(1 + 1.4) = 0.40817$$

Now let M be the number of months in the next 6 months with more than 1 accident.

A good model for the distribution of M is binomial with $m=6$ trials (each month is a trial). A month is considered to be a "success" if more than 1 accident occurs. The probability of "success" is:

$$q = \Pr(\text{"success"}) = \Pr(N \geq 2) = 0.40817$$

The probability of more than 1 success in the next 6 trials is:

$$\begin{aligned} \Pr(M \geq 2) &= 1 - \Pr(M = 0) - \Pr(M = 1) \\ &= 1 - \binom{6}{0} (0.40817)^0 (0.59183)^6 - \binom{6}{1} (0.40817)^1 (0.59183)^5 \\ &= 0.77921 \end{aligned}$$

Solution 3.4

The probability that a loss exceeds 900 is: $\Pr(X > 900) = \int_{900}^{1,000} 0.001 dx = 0.10$. Consider a loss to be a success if it exceeds 900. Due to the assumptions in the question, the losses can be viewed as being a series of independent Bernoulli trials with $q = \Pr(X > 900) = 0.10$. Let N be the number of failures observed before 2 successes are observed. Then the number of losses observed is $N + 2$, and N follows a negative binomial distribution with parameters $r = 2$ and $\beta = 0.10^{-1} - 1 = 9$. So the expected number of losses observed is:

$$E[2 + N] = 2 + r\beta = 2 + 2 \times 9 = 20$$

Solution 3.5

The given density function is for a gamma distribution with $\alpha = 3$, $\theta = 1/5$. We are given that $N | \Lambda = \lambda$ is Poisson with mean $\Lambda = \lambda$, and that Λ follows a gamma distribution. As a result, we know that N , the annual number of accidents on a randomly selected 10-mile stretch of this highway, follows a negative binomial distribution with $r = \alpha = 3$ and $\beta = \theta = 1/5$. The annual number of accidents on a 20-mile stretch of highway, M , can be viewed as a sum of 2 such negative binomial distribution. So it follows a negative binomial distribution with $r = 2 \times 3 = 6$ and $\beta = \theta = 1/5$. So the probability of $M=2$ is:

$$\Pr(M = 2) = \frac{6 \times 7}{2!} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6 = 0.19536$$

Solution 3.6

The number of consecutive road games that they will lose follows a geometric distribution with $p = 1/8$ ($\beta = 7$). We are asked to compute $E[N | N \geq 6]$. According to the memory-less property, $N - 6 | N \geq 6$ also follows this same geometric distribution. So we have:

$$\begin{aligned} E[N | N \geq 6] &= E[6 + N - 6 | N \geq 6] = 6 + E[N - 6 | N \geq 6] = 6 + E[N] \\ &= 6 + \beta = 6 + (p^{-1} - 1) = 6 + 8 - 1 = 13 \end{aligned}$$

Solution 3.7

Each of the $m=25$ lives is viewed as a Bernoulli trial. A trial is considered to be a success if the policyholder dies within 5 years. The probability of success is:

$$q = {}_5q_{50} = 1 - \frac{l_{55}}{l_{50}} = 1 - \frac{90-55}{90-50} = \frac{5}{40} \quad \text{since } l_x = 90 - x$$

The number of deaths from this group in the next 5 years follows a binomial distribution with $m=25$, $q = 1/8$. We are asked to determine:

$$\begin{aligned} \Pr(M \geq 2) &= 1 - \Pr(M=0) - \Pr(M=1) = 1 - \binom{25}{0} \left(\frac{1}{8}\right)^0 \left(\frac{7}{8}\right)^{25} - \binom{25}{1} \left(\frac{1}{8}\right)^1 \left(\frac{7}{8}\right)^{24} \\ &= 1 - 0.03550 - 0.12678 = 0.83772 \end{aligned}$$

Solution 3.8

We have a compound counting model for the annual number of payments to dependents:

$$C = M_1 + \dots + M_N \quad \text{where } N \sim \text{negative binomial } r = \beta = 2$$

From the given information it is easily checked that $E[M] = 1$, $E[M^2] = 1.6$, $\text{var}(M) = 0.6$. So from standard compound sum moment formulas, we have:

$$\begin{aligned} E[C] &= E[N] E[M] = (r\beta)1 = 4 \\ \text{var}(C) &= E[N] \text{var}(M) + (E[M])^2 \text{var}(N) \\ &= (r\beta) \times 0.6 + (1.0)^2 (r\beta(1+\beta)) = 14.4 \end{aligned}$$

Solution 3.9

We need to use Theorem 3.1:

$$r = 2, \beta = 2 \Rightarrow a = \frac{\beta}{1+\beta} = \frac{2}{3}, \quad b = \frac{(r-1)\beta}{1+\beta} = \frac{2}{3}$$

$$\text{and } P_N(z) = (1 - \beta(z-1))^{-r} = (3 - 2z)^{-2}$$

The starting value for the recursion is:

$$\Pr(C=0) = P_N(\Pr(M=0)) = P_N(0.3) = (3 - 2(0.3))^{-2} = 0.17361$$

The recursion formula is:

$$\begin{aligned}\Pr(C=n) &= \frac{1}{1 - a\Pr(M=0)} \sum_{j=1}^n \left(a + \frac{bj}{n}\right) \Pr(M=j) \Pr(C=n-j) \\ &= \frac{1}{1 - \frac{2}{3}(0.3)} \sum_{j=1}^n \left(\frac{2}{3} + \frac{2j}{3n}\right) \Pr(M=j) \Pr(C=n-j) \\ &= 1.25 \left(\left(\frac{2}{3} + \frac{2}{3n}\right) 0.4 \Pr(C=n-1) + \left(\frac{2}{3} + \frac{4}{3n}\right) 0.3 \Pr(C=n-2) \right)\end{aligned}$$

With $n=1$, we have:

$$\Pr(C=1) = 1.25 \left(\left(\frac{2}{3} + \frac{2}{3}\right) 0.4 \Pr(C=0) \right) = 0.11574$$

Finally, we have $\Pr(C \leq 1) = \Pr(C=0) + \Pr(C=1) = 0.28935$

Solution 3.10

First let $\tilde{M} = M + 1$. This variable will also count the initial earthquake in an event. The annual number of earthquakes and aftershocks is:

$$C = \tilde{M}_1 + \dots + \tilde{M}_N \quad \text{where } N \sim \text{geometric with } 2 = \beta = E[N]$$

The probability distribution of \tilde{M} is:

$$\Pr(\tilde{M}=1) = 0.10, \quad \Pr(\tilde{M}=2) = 0.60, \quad \Pr(\tilde{M}=3) = 0.30$$

The moment generating function of the geometric primary distribution is:

$$P_N(z) = (1 - \beta(z-1))^{-1} = (3 - 2z)^{-1}$$

The probability of C equal to zero is:

$$\Pr(C=0) = P_N(\Pr(\tilde{M}=0)) = P_N(0) = 3^{-1} = 1/3$$

We can use the recursion formula to calculate $\Pr(C=1)$. The compound geometric recursion formula is:

$$\begin{aligned}\Pr(C=n) &= \frac{\beta}{1 + \beta\Pr(\tilde{M} \geq 1)} \sum_{j=1}^n \Pr(\tilde{M}=j) \Pr(C=n-j) \\ &= \frac{2}{1 + 2 \times 1} (0.10 \Pr(C=n-1) + 0.60 \Pr(C=n-2) + 0.30 \Pr(C=n-3))\end{aligned}$$

So we have:

$$\Pr(C = 1) = \frac{2}{3}(0.10\Pr(C=0)) = 0.02222$$

As a result, we have:

$$\Pr(C \geq 2) = 1 - \Pr(C = 0, 1) = 1 - 0.33333 - 0.02222 = 0.64444$$

Solution 3.11

Since $N | \Lambda$ is Poisson with mean Λ , we know that $E[N | \Lambda] = \text{var}(N | \Lambda) = \Lambda$. Applying the double expectation theorem, we have:

$$\begin{aligned} E[N] &= E[E[N | \Lambda]] = E[\Lambda] = 2 \\ \text{var}(N) &= E[\text{var}(N | \Lambda)] + \text{var}(E[N | \Lambda]) = E[\Lambda] + \text{var}(\Lambda) = 2 + 2 = 4 \end{aligned}$$

Solution 3.12

We are given:

$$\begin{aligned} \Pr(N | \Lambda = \lambda) &= e^{-\lambda} \frac{\lambda^n}{n!} \text{ for } n = 0, 1, 2, \dots \\ \Pr(\Lambda = \lambda) &= e^{-2} \frac{2^\lambda}{\lambda!} \text{ for } \lambda = 0, 1, 2, \dots \end{aligned}$$

So the probability that $N = 0$ is:

$$\begin{aligned} \Pr(N = 0) &= \sum_{\lambda=0}^{\infty} \Pr(N=0 | \Lambda = \lambda) \Pr(\Lambda = \lambda) \\ &= \sum_{\lambda=0}^{\infty} e^{-\lambda} e^{-2} \frac{2^\lambda}{\lambda!} = e^{-2} \sum_{\lambda=0}^{\infty} \frac{(2e^{-1})^\lambda}{\lambda!} = e^{-2} e^{2e^{-1}} = 0.28245 \end{aligned}$$

Solution 3.13

For year 2003 we have ${}_{03}N_L$ is distributed as negative binomial where:

$$\begin{aligned} 6 &= E[{}_{03}N_L] = r_{03} \beta_{03} \text{ , } 24 = \text{var}({}_{03}N_L) = r_{03} \beta_{03} (1 + \beta_{03}) \\ \Rightarrow r_{03} &= 2 \text{ , } \beta_{03} = 3 \end{aligned}$$

From the results of Section 3.5, a 10% increase in exposure will result in ${}_{04}N_L$ following a negative binomial distribution with $r_{04} = 1.10r_{03} = 2.2$, $\beta_{04} = \beta_{03} = 3$.

From results in Chapter 2 we know that $X_{04}=1.04X_{03}$ will follow a 2-parameter Pareto distribution with parameters $\alpha_{04}=\alpha_{03}=2$ and $\theta_{04}=1.04\theta_{03}=1.04\times 500=520$. With an ordinary deductible of 100 per loss in 2004, the probability that a loss event is a payment event is:

$$\nu = \Pr(X_{04} > 100) = s_{X_{04}}(100) = \left(\frac{520}{520+100}\right)^2 = 0.70343$$

From results in Section 3.6 we know that the distribution of *claim payments* in 2004, ${}_{04}N_P$, is negative binomial with:

$$r = r_{04} = 2.2, \quad {}_p\beta_{04} = 3\nu = 2.11030$$

Solution 3.14

Since losses are fully reimburse in Year 2003, the expected annual claims payments are:

$$E[{}_{03}N_L] E[X_{03}] = 6 \times 500 = 3,000$$

The expected annual claims payments in Year 2004 can be computed in 2 different ways:

- $$E[{}_{04}N_L] E[(X_{04}-100)_+] = (2.2 \times 3)(E[X_{04}] - E[X_{04} \wedge 100])$$

$$= 6.6 \times \left(520 - 520 \left(1 - \left(\frac{520}{520+100} \right)^{2-1} \right) \right) = 2,878.45$$

Pareto: $\frac{\theta_{04}}{\alpha_{04}-1} \left(1 - \left(\frac{\theta_{04}}{\theta_{04}+d} \right)^{\alpha_{04}-1} \right)$
- $$E[{}_{04}N_P] E[\underbrace{X_{04}-100 | X > 100}_{\text{Pareto: } \alpha=2, \theta=620}] = (2.2 \times 2.11030) 620 = 2,878.45$$

The percent change is -4.052%.

Solution 3.15

The expected annual claims payments in 2003 are 3,000. The expected annual claims payments in year 2004 with an ordinary deductible of d per loss are:

$$E[{}_{04}N_L] E[(1.04X_{03}-d)_+] = E[{}_{04}N_L] \left[E[1.04X_{03}] - E \left[\underbrace{(1.04X_{03}) \wedge d}_{\substack{\text{Pareto: } \alpha=2 \\ \text{and } \theta=520}} \right] \right]$$

$$= (2.2 \times 3) \left(1.04 \times 500 - \frac{520}{2-1} \left(1 - \left(\frac{520}{520+d} \right)^{2-1} \right) \right)$$

Setting this expression equal to 3,000 results in $d = 74.88$.

Solution 3.16

In the original form we have:

$$C = M_1 + \dots + M_N \quad \text{where } N \sim \text{negative binomial } r = \beta = 2$$

$$\text{and: } \Pr(M = 0) = 0.30, \Pr(M = 1) = 0.40, \Pr(M = 2) = 0.30$$

Now let $\tilde{M} = M \mid M > 0$: $\Pr(\tilde{M} = 1) = (0.40 / 0.70)$, $\Pr(\tilde{M} = 2) = (0.30 / 0.70)$. According to the results found in Section 3.6, the frequency of non-zero terms in the original compound sum, \tilde{N} , is negative binomial with parameters:

$$r = 2, \beta^* = \beta v = 2 \Pr(M > 0) = 2 \times 0.70 = 1.4$$

In the zero-filtered form we have:

$$C = \tilde{M}_1 + \dots + \tilde{M}_{\tilde{N}}$$

Solution 3.17

From the results in Solution 3.16, we have:

$$E[C] = E[\tilde{N}] E[\tilde{M}] = (r\beta^*) \left(1 \times \frac{4}{7} + 2 \times \frac{3}{7} \right) = 4.0$$

$$\text{var}(C) = \underbrace{E[\tilde{N}]}_{r\beta^* = 2.8} \underbrace{\text{var}(\tilde{M})}_{\frac{16}{7} - \left(\frac{10}{7}\right)^2} + \underbrace{\left(E[\tilde{M}]\right)^2}_{\left(\frac{10}{7}\right)^2} \underbrace{\text{var}(\tilde{N})}_{r\beta^*(1+\beta^*) = 6.72} = 14.40$$

Solution 3.18

From the results in Solution 3.16, we have:

$$\Pr(C = 0) = \Pr(\tilde{N} = 0) = \left(\frac{1}{1 + \beta^*} \right)^r = \left(\frac{1}{2.4} \right)^2 = 0.17361$$

$$\Pr(C = 1) = \Pr(\tilde{N} = 1) \Pr(\tilde{M} = 1) = \left(\frac{2}{1} \left(\frac{\beta^*}{1 + \beta^*} \right)^1 \left(\frac{1}{1 + \beta^*} \right)^r \right) \times \frac{4}{7} = 0.11574$$

Solution 3.19

Annual claims payments are $S = X_1 \wedge 250 + \dots + X_{N_L} \wedge 250$. It is easy to compute the moments of the payment per loss variable using the formulas in Tables 2.3 and 2.4:

$$\begin{aligned} E[X \wedge 250] &= \theta(1 - e^{-250/\theta}) = 100(1 - e^{-2.5}) = 91.79 \quad (\text{See Table 2.3}) \\ E[(X \wedge 250)^2] &= 2\theta^2 \Gamma(3; 250/\theta) + 250^2(e^{-250/\theta}) \\ &= 20,000 \left(1 - e^{-2.5} \left(1 + 2.5 + \frac{2.5^2}{2!} \right) \right) + 250^2 e^{-2.5} = 14,254.05010 \end{aligned}$$

Since the frequency model is Poisson with mean (and variance equal to 20), we have:

$$\begin{aligned} E[S] &= 20 E[X \wedge 250] = 1,835.83 \\ \text{var}(S) &= 20 E[(X \wedge 250)^2] = 285,081 \end{aligned}$$

Solution 3.20

The first step is to determine d such that:

$$E[(X - d)_+] = E[X \wedge 250] = 91.79150$$

Form an exponential distribution formula in Table 2.3, we have:

$$91.79150 = E[(X - d)_+] = \theta e^{-d/\theta} = 100 e^{-d/100} \Rightarrow d = 8.56505$$

Since the frequency of losses is Poisson, the variance in annual claims payments is:

$$\begin{aligned} \text{var}(S) &= 20 E[(X - d)_+^2] = 20 \times \underbrace{E[(X - d)^2 | X > d]}_{\substack{\text{conditional exponential} \\ \text{second moment is } 2(100)^2}} \underbrace{\Pr(X > d)}_{\substack{e^{-d/100} \\ = 0.91792}} \\ &= 367,166 \end{aligned}$$