



# Construction of Actuarial Models

Second Edition

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## Solutions to practice questions – Chapter 4

### Solution 4.1

According to Theorem 4.1, the new severity model is a weighted average of the two component severity models:

$$f_X(x) = \frac{15}{20} f_1(x) + \frac{5}{20} f_2(x)$$

where:

$$f_1(x) = 0.001 \quad \text{for } 0 < x \leq 1,000 \quad (\text{zero otherwise})$$

$$f_2(x) = \frac{3(2,000)^3}{(2,000 + x)^4} \quad \text{for } x > 0 \quad (\text{zero otherwise})$$

### Solution 4.2

The expected loss is:

$$\begin{aligned} E[X] &= \frac{15}{20} E[X_1] + \frac{5}{20} E[X_2] \quad \text{where } X_1 \sim U(0, 1,000), X_2 \sim \text{Pareto } \alpha = 3, \theta = 2,000 \\ &= \frac{15}{20} \times \frac{1,000}{2} + \frac{5}{20} \times \frac{2,000}{3-1} = 625 \end{aligned}$$

So the probability that a loss exceeds the expected value is:

$$\begin{aligned} \Pr(X > 625) &= \int_{625}^{\infty} f_X(x) dx = \int_{625}^{\infty} (0.75 f_{X_1}(x) + 0.25 f_{X_2}(x)) dx \\ &= 0.75 \int_{625}^{\infty} f_{X_1}(x) dx + 0.25 \int_{625}^{\infty} f_{X_2}(x) dx \\ &= 0.75 s_{X_1}(625) + 0.25 s_{X_2}(625) \\ &= 0.75 \left( 1 - \frac{625}{1,000} \right) + 0.25 \left( \frac{2,000}{2,000 + 625} \right)^3 = 0.39182 \end{aligned}$$

**Solution 4.3**

The expected aggregate annual loss is:

$$E[S] = E[N_1] E[X_1] + E[N_2] E[X_2] = 15 \times \frac{1,000}{2} + 5 \times \frac{2,000}{3-1} = 12,500$$

It could also have been computed using the result of Question 4.2:  $E[S] = E[N] E[X] = 20 \times 625 = 12,500$ . The compound Poisson variance formula from Theorem 4.1 results in:

$$\text{var}(S) = \lambda_1 E[X_1^2] + \lambda_2 E[X_2^2] = 15 \times \frac{1,000^2}{3} + 5 \times \frac{2,000^2 \times 2!}{(3-1)(2-1)} = 25 \text{ million}$$

Approximating the distribution of  $S$  by a normal distribution with  $\mu = 12,500$ ,  $\sigma = 5,000$ , we have:

$$\begin{aligned} \Pr(S \geq 1.5E[S]) &= \Pr(S \geq 18,750) \approx 1 - \Phi\left(\frac{18,750 - 12,500}{5,000}\right) = 1 - \Phi(1.25) \\ &\approx 1 - \frac{0.8849 + 0.9032}{2} = 0.106 \end{aligned}$$

**Solution 4.4**

For the lognormal approximation, we assume that  $S \approx L = e^{N(\mu, \sigma^2)}$  where:

$$\begin{aligned} 12,500 &= E[S] = E[L] = e^{\mu + 0.5\sigma^2} \\ 25,000,000 + 12,500^2 &= E[S^2] = E[L^2] = e^{2\mu + 2\sigma^2} \end{aligned}$$

Square the first equation, take natural log of both sides of both equations, and then subtract the first equation from the second. The result is:

$$\sigma^2 = 0.14842, \quad \mu = 9.35927$$

Approximating the distribution of  $S$  by a lognormal distribution with  $\sigma^2 = 0.14842$ ,  $\mu = 9.35927$ , we have:

$$\begin{aligned} \Pr(S \geq 1.5E[S]) &= \Pr(S \geq 18,750) \approx \Pr\left(N(\mu, \sigma^2) \geq \ln(18,750) = 9.83895\right) \\ &= 1 - \Phi\left(\frac{9.83895 - \mu}{\sigma}\right) = 1 - \Phi(1.24509) \end{aligned}$$

So the answer is virtually the same as in Solution 4.3.

**Solution 4.5**

From Theorem 2.1, we have:

$$250 = E[Y] = E[Z] \underbrace{\Pr(Y > 0)}_{0.80} \Rightarrow E[Z] = 312.50 \text{ (the expected payment per claim)}$$

From the relation  $S = Y_1 + \dots + Y_{N_L} = Z_1 + \dots + Z_{N_P}$ , we also have:

$$\underbrace{E[N_L]}_{10} \underbrace{E[Y]}_{250} = E[S] = E[N_P] \underbrace{E[Z]}_{312.50} \Rightarrow E[N_P] = 8 \text{ (the expected annual number of claims)}$$


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**Solution 4.6**

Aggregate annual claims follow a compound Poisson distribution:

$$S = (X_1 - 100)_+ + \dots + (X_{N_L} - 100)_+ \text{ where } N_L \sim \text{Poisson } \lambda = 10$$

The severity model  $X$  follows an exponential distribution with mean  $\theta = 500$ . From Tables 2.3 and 2.4, we have:

$$E[X \wedge d] = \theta(1 - e^{-d/\theta}) = 500(1 - e^{-100/500}) = 90.63462 \quad (\text{Table 2.3})$$

$$\begin{aligned} E[(X \wedge d)^2] &= 2\theta^2 \Gamma(3; d/\theta) + d^2 e^{-d/\theta} && (\text{Table 2.4}) \\ &= 2(500)^2 \Gamma(3; 0.2) + 100^2 e^{-0.2} \\ &= 500,000 \left( 1 - e^{-0.2} \left( 1 + 0.2 + \frac{0.2^2}{2!} \right) \right) + 8,187.30753 = 8,761.54813 \end{aligned}$$

From Theorem 2.2 and Theorem 2.4(ii), we have:

$$\begin{aligned} E[(X - 100)_+] &= E[X] - E[X \wedge 100] = 500 - 90.63462 = 409.36538 \\ E[(X - 100)_+^2] &= E[X^2] - E[(X \wedge 100)^2] - 2 \times 100 E[(X - 100)_+] \\ &= 2(500)^2 - 8,761.54813 - 200 \times 409.36538 = 409,365.3766 \end{aligned}$$

These moments could have been calculated more quickly by realizing that  $Z = X - 100 \mid X > 100$  is also exponentially distributed with parameter  $\theta = 500$  (see Table 2.2):

$$\begin{aligned} E[(X - 100)_+] &= E[Z] \Pr(Y > 0) = 500 \times e^{-100/500} = 409.36538 \\ E[(X - 100)_+^2] &= E[Z^2] \Pr(Y > 0) = 2 \times 500^2 \times e^{-100/500} = 409,365.3766 \end{aligned}$$

So from Theorem 4.1, we have:

$$E[S] = \lambda E[(X - 100)_+] = 4,093.65 \quad , \quad \text{var}(S) = \lambda E[(X - 100)_+^2] = 4,093,653.77$$

**Solution 4.7**

We assume that  $S$  is approximately normal in distribution with mean  $\mu = 4,093.65$  and variance  $\sigma^2 = 4,093,653.77$ . Now we need the expected limited loss formula for a normal distribution:

$$\begin{aligned} E[S \wedge d] &= (\mu - d) \Phi\left(\frac{d - \mu}{\sigma}\right) + d - \frac{\sigma}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{d - \mu}{\sigma}\right)^2\right] \\ &= (4,093.65 - 7,500) \Phi\left(\frac{7,500 - 4,093.65}{\sqrt{4,093,653.77}}\right) + 7,500 - \sqrt{\frac{4,093,653.77}{2\pi}} \times \exp\left(-\frac{(7,500 - 4,093.65)^2}{2 \times 4,093,653.77}\right) \\ &= -3,406.35 \underbrace{\Phi(1.68)}_{\approx 0.9534} + 7,500 - 807.17 \times 0.24239 = 4,056.74 \\ E[(S - 7,500)_+] &= E[S] - E[S \wedge 7,500] = 36.91 \end{aligned}$$


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**Solution 4.8**

The lognormal approximation takes a little more effort:

$$\begin{aligned} 4,093.65 &= E[S] = e^{\mu + 0.5\sigma^2} \quad , \quad 20,851,655 = E[S^2] = e^{2\mu + 2\sigma^2} \\ \Rightarrow \sigma^2 &= 0.21856 \quad , \quad \mu = 8.20791 \end{aligned}$$

Now we need the expected limited loss formula for the lognormal distribution:

$$\begin{aligned} E[S \wedge d] &= e^{\mu + 0.5\sigma^2} \Phi\left(\frac{\log(d) - \mu - \sigma^2}{\sigma}\right) + d \left(1 - \Phi\left(\frac{\log(d) - \mu}{\sigma}\right)\right) \\ &= 4,093.65 \Phi\left(\frac{8.92266 - 8.20791 - 0.21856}{\sqrt{0.21856}}\right) + 7,500 \left(1 - \Phi\left(\frac{8.92266 - 8.20791}{\sqrt{0.21856}}\right)\right) \\ &= 4,093.65 \underbrace{\Phi(1.061)}_{0.8553} + 7,500 \left(1 - \underbrace{\Phi(1.529)}_{0.9369}\right) = 3,976.32 \\ E[(S - 7,500)_+] &= E[S] - E[S \wedge 7,500] = 117.32 \end{aligned}$$


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**Solution 4.9**

From Theorem 3.2 and the table following this theorem, we know that  $N_p$  follows a Poisson distribution with parameter  $\lambda^*$  that is closely related to the parameter  $\lambda$ :

$$\begin{aligned} \nu &= \Pr(X > d) = s_X(d) = s_X(100) = e^{-100/500} = 0.81873 \\ \lambda^* &= \lambda\nu = 10 \times 0.81873 = 8.18731 \end{aligned}$$

**Solution 4.10**

If  $n$  is the number of policies in Year 2004, and  $n_1$  is the number of policies in Year 2005, we are given that  $1.10 = n_1/n$ , a 10% increase in exposure. From results in Section 5 of Chapter 3, it follows that the model for the frequency of losses in Year 2005,  $N_{05:L}$ , is Poisson with parameter:

$$\lambda^* = n_1 \lambda / n = 1.10 \times 10 = 11$$


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**Solution 4.11**

The loss model in Year 2005 is  $X_{05} = 1.03X$ . Since the parameter  $\theta$  in an exponential distribution is a scale parameter, it follows that  $X_{05}$  follows an exponential distribution with mean  $\theta^* = 1.03\theta = 515$ . We have seen in Solution 4.10 that the frequency of losses in Year 2005,  $N_{05:L}$ , is Poisson with parameter  $\lambda^* = 11$ .

From Theorem 3.2 and the following table in Section 6 of Chapter 3, we know that the frequency of payment events in Year 2005 is Poisson with parameter:

$$\lambda^{**} = \nu \lambda^* = \Pr(X_{05} > 100) \times 11 = e^{-100/515} \times 11 = 9.05865$$


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**Solution 4.12**

The claim payment in Year 2004 is  $X - 100 \mid X > 100$ , which follows the same distribution as  $X$ , exponential with mean 500. The claim payment in Year 2005 is  $X_{05} - 100 \mid X_{05} > 100$ , which follows the same distribution as  $X_{05}$ , exponential with mean 515.

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**Solution 4.13**

In Solution 4.6 we calculated the expected annual claims payments in Year 2004 as:

$$E[S] = \lambda E[(X - 100)_+] = 4,093.65$$

If there is an ordinary deductible of  $d$  per loss, then the expected annual claims payments in Year 2005 are:

$$\begin{aligned} E[S_{05}] &= E[N_{05:L}] E[(X_{05} - d)_+] = \underbrace{E[N_{05:L}]}_{\text{Solution 12.10}} \times \underbrace{E[X_{05} - d \mid X_{05} > d]}_{\substack{\text{Same mean as } X_{05} \text{ since} \\ \text{amounts are exponentially} \\ \text{distributed}}} \times \Pr(X_{05} > d) \\ &= 11 \times 515 \times e^{-d/515} \end{aligned}$$

Equating these expected values and solving for  $d$  results in  $d = 167.31$

**Solution 4.14**

The payment per loss in Year 2005 would be:

$$Y_{05} = \begin{cases} 0 & \text{if } X_{05} \leq 100 \\ X_{05} - 100 & \text{if } 100 < X_{05} < 100 + L \\ L & \text{if } 100 + L \leq X_{05} \end{cases}$$

$$= X_{05} \wedge (100 + L) - X_{05} \wedge 100$$

Using the expected limited loss formula for an exponential distribution, we have:

$$E[X_{05} \wedge x] = \theta_{05} (1 - e^{-x/\theta_{05}}) = 515 (1 - e^{-x/515}) \Rightarrow$$

$$E[Y_{05}] = E[X_{05} \wedge (100 + L)] - E[X_{05} \wedge 100]$$

$$= 515 \left( 1 - e^{-(100+L)/515} \right) - 515 \left( 1 - e^{-100/515} \right)$$

$$= 424.10970 \left( 1 - e^{-L/515} \right)$$

To hold expected annual claims payments in Year 2005 to the same level as in Year 2004, we would have:

$$4,093.65 = E[S_{04}] = E[S_{05}] = E[N_{05:L}] E[Y_{05}] = 11 \times 424.10970 \left( 1 - e^{-L/515} \right)$$

$$\Rightarrow e^{-L/515} = 0.12251 \Rightarrow L = 1,081.25$$


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**Solution 4.15**

In Solution 4.6 when it was assumed that there was an ordinary deductible of 100 per loss, we calculated the expected value and variance of the annual claims payments as:

$$E[S] = \lambda E[(X - 100)_+] = 4,093.65 \quad , \quad \text{var}(S) = \lambda E[(X - 100)_+^2] = 4,093,653.77$$

If the deductible amount is replaced by a policy limit that results in the same level of expected annual claims, we have:

$$409.36538 = E[(X - 100)_+] = E[X \wedge L] = 500 \left( 1 - e^{-L/500} \right) \Rightarrow L = 853.88590$$

With this policy limit, the variance in annual claims payments is determined as follows:

$$\begin{aligned}
 E[(X \wedge L)^2] &= 2\theta^2 \Gamma(3; L/\theta) + L^2 e^{-L/\theta} \quad (\text{Table 2.4}) \\
 &= 500,000 \left( 1 - e^{-1.70777} \left( 1 + 1.70777 + \frac{1.70777^2}{2!} \right) \right) + 853.88590^2 e^{-1.70777} \\
 &= 122,414.8843 + 132,167.2383 = 254,582.1226 \\
 \Rightarrow \text{var}(S) &= \lambda E[(X \wedge L)^2] = 10 \times 254,582.1226 = 2,545,821.23
 \end{aligned}$$

This is 62.2% of the variance with a deductible rather than a limit.

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### Solution 4.16

The distribution of  $Y = (X - 1)_+$  is:

$$\Pr(Y=0) = 0.7 \quad , \quad \Pr(Y=1) = 0.2 \quad , \quad \Pr(Y=2) = 0.1$$

Since  $N$  is Poisson distributed with mean  $\lambda = 4$ , the recursion formula and the starting value (Section 3.3) are:

$$\begin{aligned}
 \Pr(S=0) &= P_N(\Pr(Y=0)) = e^{\lambda(\Pr(Y=0)-1)} = e^{4(0.7-1)} = e^{-1.2} \\
 \lambda_i &= \lambda \Pr(Y=i) \Rightarrow \lambda_1 = 0.8 \quad , \quad \lambda_2 = 0.4 \Rightarrow \\
 \Pr(S=n) &= \frac{1}{n} (1 \cdot \lambda_1 \cdot \Pr(S=n-1) + 2 \cdot \lambda_2 \cdot \Pr(S=n-2)) \\
 &= \frac{1}{n} (0.8 \Pr(S=n-1) + 0.8 \Pr(S=n-2))
 \end{aligned}$$

Use this formula successively with  $n=1, 2$ , and  $3$ :

$$\begin{aligned}
 \Pr(S=1) &= 0.8 \Pr(S=0) = 0.8 e^{-1.2} \\
 \Pr(S=2) &= \frac{1}{2} (0.8 \Pr(S=1) + 0.8 \Pr(S=0)) = 0.72 e^{-1.2} \\
 \Pr(S=3) &= \frac{1}{3} (0.8 \Pr(S=2) + 0.8 \Pr(S=1)) = 0.40533 e^{-1.2}
 \end{aligned}$$

Totaling these 4 probabilities you will find that:  $\Pr(S \leq 3) = 2.92533 e^{-1.2} = 0.88109$ .

**Solution 4.17**

In order to calculate from combinatorial reasoning, we must first filter out the zero terms in the sum and then adjust the frequency distribution (see Section 3.7). Let's set notation:

$$S = Y_1 + \dots + Y_{N_L} \quad \text{where } N_L \sim \text{Poisson } \lambda = 4$$

$$= Z_1 + \dots + Z_{N_P} \quad \text{where } Z = Y|Y > 0 \text{ and } N_P \sim \text{Poisson } \lambda^* = \lambda \Pr(Y > 0)$$

From Solution 4.16, we have:

$$\Pr(Z = 1) = \frac{\Pr(Y = 1)}{\Pr(Y > 0)} = \frac{2}{3}, \quad \Pr(Z = 2) = \frac{\Pr(Y = 2)}{\Pr(Y > 0)} = \frac{1}{3}, \quad \lambda^* = 4 \times 0.3 = 1.2$$

Now it is easy to duplicate the probability calculations in Solution 4.16. For example, we have:

$$\Pr(S = 0) = \Pr(N_P = 0) = e^{-\lambda^*} = e^{-1.2}$$

$$\Pr(S = 1) = \Pr(N_P = 1 \text{ and } Z=1) = e^{-1.2} \frac{1.2^1}{1!} \times \frac{2}{3} = 0.8e^{-1.2}$$

It is left to the reader to verify the calculations for  $\Pr(S = 2)$  and  $\Pr(S = 3)$ .

**Solution 4.18**

$$E[(S - 3.8)_+] = E[S] - E[S \wedge 3.8]$$

$$= \lambda E[(X - 1)_+] - (0 \cdot \Pr(S=0) + 1 \cdot \Pr(S=1) + 2 \cdot \Pr(S=2) + 3 \cdot \Pr(S=3) + 3.8 \times \Pr(S \geq 4))$$

$$= 4 \times 0.4 - (0.8e^{-1.2} + 2 \times 0.72e^{-1.2} + 3 \times 0.40533e^{-1.2} + 3.8 \times 0.11891) = 0.10723$$

**Solution 4.19**

Aggregate annual claims follow a compound Poisson distribution:

$$S = (X_1 - 100)_+ + \dots + (X_{N_L} - 100)_+ \quad \text{where } N_L \sim \text{Poisson } \lambda = 10$$

The severity model  $X$  follows an exponential distribution with mean  $\theta = 500$ . For each loss  $Y = (X - 100)_+$  of the insurer, the reinsurer pays the insurer an amount  $R$  equal to the excess of  $Y$  over 500, if there is an excess:

$$R = (Y - 500)_+ = ((X - 100)_+ - 500)_+ = (X - 600)_+$$



The total payment by the reinsurer to the insurer is thus  $S_{\text{re}} = R_1 + \dots + R_{N_L}$ . The pure reinsurance premium is the expected value of  $S_{\text{re}}$ . Since  $X$  follows an exponential distribution with mean  $\theta=500$ , we know that  $X - 600 \mid X > 600$  follows this same exponential distribution. As a result, we have:

$$E[R] = E[(X - 600)_+] = E[X - 600 \mid X > 600] \Pr(X > 600) = 500 \times e^{-600/500} = 150.59711$$

$$E[S_{\text{re}}] = E[N_L] E[R] = 10 \times 150.59711 = 1,505.97$$

### **Solution 4.20**

Aggregate annual claims follow a compound Poisson distribution:

$$S = (X_1 - 100)_+ + \dots + (X_{N_L} - 100)_+ \quad \text{where } N_L \sim \text{Poisson } \lambda = 10$$

The severity model  $X$  follows an exponential distribution with mean  $\theta=500$ . For each loss  $Y = (X - 100)_+$  of the insurer, the reinsurer pays the insurer an amount  $R = 0.25 Y$ . In this case it is easy to see that the total reinsurance payments are  $S_{\text{Re}} = 0.25 S$ . So the pure reinsurance premium is:

$$E[S_{\text{Re}}] = 0.25 E[S] = 0.25 \times \underbrace{4,093.65377}_{\text{Solution 4.6}} = 1,023.41$$