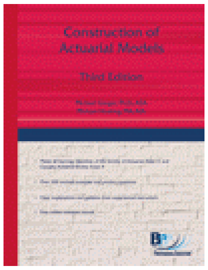

Construction of Actuarial Models

Third Edition

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Solutions to practice questions – Chapter 4

Solution 4.1

According to Theorem 4.1, the new severity model is a weighted average of the two component severity models:

$$f_X(x) = \frac{15}{20} f_1(x) + \frac{5}{20} f_2(x)$$

where:

$$f_1(x) = 0.001 \quad \text{for} \quad 0 < x \le 1{,}000 \quad \text{(zero otherwise)}$$

$$f_2(x) = \frac{3(2{,}000)^3}{(2{,}000 + x)^4} \quad \text{for} \ x > 0 \qquad \text{(zero otherwise)}$$

Solution 4.2

The expected loss is:

$$E[X] = \frac{15}{20} E[X_1] + \frac{5}{20} E[X_2] \quad \text{where } X_1 \sim U(0{,}1{,}000), \ X_2 \sim \text{Pareto } \alpha = 3, \ \theta = 2{,}000$$

$$= \frac{15}{20} \times \frac{1{,}000}{2} + \frac{5}{20} \times \frac{2{,}000}{3-1} = 625$$

So the probability that a loss exceeds the expected value is:

$$\Pr(X > 625) = \int_{625}^{\infty} f_X(x)\,dx = \int_{625}^{\infty} \left(0.75 f_{X_1}(x) + 0.25 f_{X_2}(x)\right) dx$$

$$= 0.75 \int_{625}^{\infty} f_{X_1}(x)\,dx + 0.25 \int_{625}^{\infty} f_{X_2}(x)\,dx$$

$$= 0.75\, s_{X_1}(625) + 0.25\, s_{X_2}(625)$$

$$= 0.75\left(1 - \frac{625}{1{,}000}\right) + 0.25\left(\frac{2{,}000}{2{,}000+625}\right)^3 = 0.39182$$

Solution 4.3

The expected aggregate annual loss is:

$$E[S] = E[N_1]\,E[X_1] + E[N_2]\,E[X_2] = 15 \times \frac{1{,}000}{2} + 5 \times \frac{2{,}000}{3-1} = 12{,}500$$

It could also have been computed using the result of Question 4.2: $E[S] = E[N]\,E[X] = 20 \times 625 = 12{,}500$. The compound Poisson variance formula from Theorem 4.1 results in:

$$\text{var}(S) = \lambda_1\,E\left[X_1^2\right] + \lambda_2\,E\left[X_2^2\right] = 15 \times \frac{1{,}000^2}{3} + 5 \times \frac{2{,}000^2 \times 2!}{(3-1)(2-1)} = 25 \text{ million}$$

Approximating the distribution of S by a normal distribution with $\mu = 12{,}500$, $\sigma = 5{,}000$, we have:

$$\Pr(S \geq 1.5E[S]) = \Pr(S \geq 18{,}750) \approx 1 - \Phi\left(\frac{18{,}750 - 12{,}500}{5{,}000}\right) = 1 - \Phi(1.25)$$

$$\approx 1 - \frac{0.8849 + 0.9032}{2} = 0.106$$

Solution 4.4

For the lognormal approximation, we assume that $S \approx L = e^{N(\mu,\sigma^2)}$ where:

$$12{,}500 = E[S] = E[L] = e^{\mu + 0.5\sigma^2}$$

$$25{,}000{,}000 + 12{,}500^2 = E\left[S^2\right] = E\left[L^2\right] = e^{2\mu + 2\sigma^2}$$

Square the first equation, take natural log of both sides of both equations, and then subtract the first equation from the second. The result is:

$$\sigma^2 = 0.14842 \ , \ \mu = 9.35927$$

Approximating the distribution of S by a lognormal distribution with $\sigma^2 = 0.14842$, $\mu = 9.35927$, we have:

$$\Pr(S \geq 1.5E[S]) = \Pr(S \geq 18{,}750) \approx \Pr\left(N\left(\mu,\sigma^2\right) \geq \ln(18{,}750) = 9.83895\right)$$

$$= 1 - \Phi\left(\frac{9.83895 - \mu}{\sigma}\right) = 1 - \Phi(1.24509)$$

So the answer is virtually the same as in Solution 4.3.

Solution 4.5

From Theorem 2.1, we have:

$$250 = E[Y] = E[Z]\underbrace{\Pr(Y>0)}_{0.80} \;\Rightarrow\; E[Z] = 312.50 \text{ (the expected payment per claim)}$$

From the relation $S = Y_1 + \cdots + Y_{N_L} = Z_1 + \cdots + Z_{N_P}$, we also have:

$$\underbrace{E[N_L]}_{10}\,\underbrace{E[Y]}_{250} = E[S] = E[N_P]\underbrace{E[Z]}_{312.50} \;\Rightarrow\; E[N_P] = 8 \text{ (the expected annual number of claims)}$$

Solution 4.6

Aggregate annual claims follow a compound Poisson distribution:

$$S = (X_1 - 100)_+ + \cdots + \left(X_{N_L} - 100\right)_+ \quad \text{where } N_L \sim \text{Poisson } \lambda = 10$$

The severity model X follows an exponential distribution with mean $\theta = 500$. From Tables 2.3 and 2.4, we have:

$$E[X \wedge d] = \theta\left(1 - e^{-d/\theta}\right) = 500\left(1 - e^{-100/500}\right) = 90.63462 \qquad \text{(Table 2.3)}$$

$$E\left[(X \wedge d)^2\right] = 2\theta^2\Gamma(3\,;\,d/\theta) + d^2 e^{-d/\theta} \qquad \text{(Table 2.4)}$$

$$= 2(500)^2\,\Gamma(3\,;\,0.2) + 100^2 e^{-0.2}$$

$$= 500{,}000\left(1 - e^{-0.2}\left(1 + 0.2 + \frac{0.2^2}{2!}\right)\right) + 8{,}187.30753 = 8{,}761.54813$$

From Theorem 2.2 and Theorem 2.4(ii), we have:

$$E\left[(X-100)_+\right] = E[X] - E[X \wedge 100] = 500 - 90.63462 = 409.36538$$

$$E\left[(X-100)_+^2\right] = E\left[X^2\right] - E\left[(X \wedge 100)^2\right] - 2\times 100\,E\left[(X-100)_+\right]$$

$$= 2(500)^2 - 8{,}761.54813 - 200 \times 409.36538 = 409{,}365.3766$$

These moments could have been calculated more quickly by realizing that $Z = X - 100 \mid X > 100$ is also exponentially distributed with parameter $\theta = 500$ (see Table 2.2):

$$E\left[(X-100)_+\right] = E[Z]\Pr(Y>0) = 500 \times e^{-100/500} = 409.36538$$

$$E\left[(X-100)_+^2\right] = E\left[Z^2\right]\Pr(Y>0) = 2 \times 500^2 \times e^{-100/500} = 409{,}365.3766$$

So from Theorem 4.1, we have:

$$E[S] = \lambda E\left[(X-100)_+\right] = 4{,}093.65 \quad , \quad \text{var}(S) = \lambda E\left[(X-100)_+^2\right] = 4{,}093{,}653.77$$

Solution 4.7

We assume that S is approximately normal in distribution with mean $\mu = 4{,}093.65$ and variance $\sigma^2 = 4{,}093{,}653.77$. Now we need the expected limited loss formula for a normal distribution:

$$E[S \wedge d] = (\mu - d)\ \Phi\left(\frac{d-\mu}{\sigma}\right) + d - \frac{\sigma}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}\left(\frac{d-\mu}{\sigma}\right)^2\right]$$

$$= (4{,}093.65 - 7{,}500)\Phi\left(\frac{7{,}500 - 4{,}093.65}{\sqrt{4{,}093{,}653.77}}\right) + 7{,}500 - \sqrt{\frac{4{,}093{,}653.77}{2\pi}} \times \exp\left(-\frac{(7{,}500 - 4{,}093.65)^2}{2 \times 4{,}093{,}653.77}\right)$$

$$= -3{,}406.35\ \underbrace{\Phi(1.68)}_{\approx 0.9534} + 7{,}500 - 807.17 \times 0.24239 = 4{,}056.74$$

$$E\left[(S - 7{,}500)_+\right] = E[S] - E[S \wedge 7{,}500] = 36.91$$

Solution 4.8

The lognormal approximation takes a little more effort:

$$4{,}093.65 = E[S] = e^{\mu + 0.5\sigma^2} \quad , \quad 20{,}851{,}655 = E\left[S^2\right] = e^{2\mu + 2\sigma^2}$$

$$\Rightarrow \quad \sigma^2 = 0.21856 \quad , \quad \mu = 8.20791$$

Now we need the expected limited loss formula for the lognormal distribution:

$$E[S \wedge d] = e^{\mu + 0.5\sigma^2}\ \Phi\left(\frac{\log(d) - \mu - \sigma^2}{\sigma}\right) + d\left(1 - \Phi\left(\frac{\log(d) - \mu}{\sigma}\right)\right)$$

$$= 4{,}093.65\ \Phi\left(\frac{8.92266 - 8.20791 - 0.21856}{\sqrt{0.21856}}\right) + 7{,}500\left(1 - \Phi\left(\frac{8.92266 - 8.20791}{\sqrt{0.21856}}\right)\right)$$

$$= 4{,}093.65\ \underbrace{\Phi(1.061)}_{0.8553} + 7{,}500\left(1 - \underbrace{\Phi(1.529)}_{0.9369}\right) = 3{,}976.32$$

$$E\left[(S - 7{,}500)_+\right] = E[S] - E[S \wedge 7{,}500] = 117.32$$

Solution 4.9

From Theorem 3.2 and the table following this theorem, we know that N_P follows a Poisson distribution with parameter λ^* that is closely related to the parameter λ:

$$\upsilon = \Pr(X > d) = s_X(d) = s_X(100) = e^{-100/500} = 0.81873$$

$$\lambda^* = \lambda\upsilon = 10 \times 0.81873 = 8.18731$$

Solution 4.10

If n is the number of policies in Year 2004, and n_1 is the number of policies in Year 2005, we are given that $1.10 = n_1 / n$, a 10% increase in exposure. From results in Section 5 of Chapter 3, it follows that the model for the frequency of losses in Year 2005, $N_{05:L}$, is Poisson with parameter:

$$\lambda^* = n_1 \lambda / n = 1.10 \times 10 = 11$$

Solution 4.11

The loss model in Year 2005 is $X_{05} = 1.03X$. Since the parameter θ in an exponential distribution is a scale parameter, it follows that X_{05} follows an exponential distribution with mean $\theta^* = 1.03\theta = 515$. We have seen in Solution 4.10 that the frequency of losses in Year 2005, $N_{05:L}$ is Poisson with parameter $\lambda^* = 11$.
From Theorem 3.2 and the following table in Section 6 of Chapter 3, we know that the frequency of payment events in Year 2005 is Poisson with parameter:

$$\lambda^{**} = \upsilon\lambda^* = \Pr\left(X_{05} > 100\right) \times 11 = e^{-100/515} \times 11 = 9.05865$$

Solution 4.12

The claim payment in Year 2004 is $X - 100 \mid X > 100$, which follows the same distribution as *X*, exponential with mean 500. The claim payment in Year 2005 is $X_{05} - 100 \mid X_{05} > 100$, which follows the same distribution as X_{05}, exponential with mean 515.

Solution 4.13

In Solution 4.6 we calculated the expected annual claims payments in Year 2004 as:

$$E[S] = \lambda E\left[(X - 100)_+\right] = 4{,}093.65$$

If there is an ordinary deductible of *d* per loss, then the expected annual claims payments in Year 2005 are:

$$E[S_{05}] = E[N_{05:L}]E\left[(X_{05} - d)_+\right] = \underbrace{E[N_{05:L}]}_{\text{Solution 12.10}} \times \underbrace{E\left[X_{05} - d \mid X_{05} > d\right]}_{\substack{\text{Same mean as } X_{05} \text{ since} \\ \text{amounts are exponentially} \\ \text{distributed}}} \times \Pr\left(X_{05} > d\right)$$

$$= 11 \times 515 \times e^{-d/515}$$

Equating these expected values and solving for *d* results in $d = 167.31$

Solution 4.14

The payment per loss in Year 2005 would be:

$$Y_{05} = \begin{cases} 0 & \text{if } X_{05} \le 100 \\ X_{05} - 100 & \text{if } 100 < X_{05} < 100 + L \\ L & \text{if } 100 + L \le X_{05} \end{cases}$$

$$= X_{05} \wedge (100 + L) \ - \ X_{05} \wedge 100$$

Using the expected limited loss formula for an exponential distribution, we have:

$$E[X_{05} \wedge x] = \theta_{05}\left(1 - e^{-x/\theta_{05}}\right) = 515\left(1 - e^{-x/515}\right) \ \Rightarrow$$

$$E[Y_{05}] = E[X_{05} \wedge (100 + L)] - E[X_{05} \wedge 100]$$

$$= 515\left(1 - e^{-(100+L)/515}\right) - 515\left(1 - e^{-100/515}\right)$$

$$= 424.10970\left(1 - e^{-L/515}\right)$$

To hold expected annual claims payments in Year 2005 to the same level as in Year 2004, we would have:

$$4{,}093.65 = E[S_{04}] = E[S_{05}] = E[N_{05:L}]\, E[Y_{05}] = 11 \times 424.10970\left(1 - e^{-L/515}\right)$$

$$\Rightarrow \ e^{-L/515} = 0.12251 \ \Rightarrow \ L = 1{,}081.25$$

Solution 4.15

In Solution 4.6 when it was assumed that there was an ordinary deductible of 100 per loss, we calculated the expected value and variance of the annual claims payments as:

$$E[S] = \lambda E\left[(X - 100)_+\right] = 4{,}093.65 \quad , \quad \text{var}(S) = \lambda E\left[(X - 100)_+^2\right] = 4{,}093{,}653.77$$

If the deductible amount is replaced by a policy limit that results in the same level of expected annual claims, we have:

$$409.36538 = E\left[(X - 100)_+\right] = E[X \wedge L] = 500\left(1 - e^{-L/500}\right) \ \Rightarrow \ L = 853.88590$$

With this policy limit, the variance in annual claims payments is determined as follows:

$$E\left[(X \wedge L)^2\right] = 2\theta^2\,\Gamma(3\,;L/\theta) + L^2\,e^{-L/\theta} \qquad \text{(Table 2.4)}$$

$$= 500{,}000\left(1 - e^{-1.70777}\left(1 + 1.70777 + \frac{1.70777^2}{2!}\right)\right) + 853.88590^2\,e^{-1.70777}$$

$$= 122{,}414.8843 + 132{,}167.2383 = 254{,}582.1226$$

$$\Rightarrow \quad \operatorname{var}(S) = \lambda\,E\left[(X \wedge L)^2\right] = 10 \times 254{,}582.1226 = 2{,}545{,}821.23$$

This is 62.2% of the variance with a deductible rather than a limit.

Solution 4.16

The distribution of $Y = (X-1)_+$ is:

$$\Pr(Y=0) = 0.7 \quad , \quad \Pr(Y=1) = 0.2 \quad , \quad \Pr(Y=2) = 0.1$$

Since *N* is Poisson distributed with mean $\lambda = 4$, the recursion formula and the starting value (Section 3.3) are:

$$\Pr(S=0) = P_N\left(\Pr(Y=0)\right) = e^{\lambda(\Pr(Y=0)-1)} = e^{4(0.7-1)} = e^{-1.2}$$

$$\lambda_i = \lambda\Pr(Y=i) \Rightarrow \lambda_1 = 0.8\ ,\ \lambda_2 = 0.4 \Rightarrow$$

$$\Pr(S=n) = \frac{1}{n}\left(1\cdot\lambda_1\cdot\Pr(S=n-1) + 2\cdot\lambda_2\cdot\Pr(S=n-2)\right)$$

$$= \frac{1}{n}\left(0.8\Pr(S=n-1) + 0.8\Pr(S=n-2)\right)$$

Use this formula successively with $n = 1$, 2, and 3:

$$\Pr(S=1) = 0.8\Pr(S=0) = 0.8e^{-1.2}$$

$$\Pr(S=2) = \frac{1}{2}\left(0.8\Pr(S=1) + 0.8\Pr(S=0)\right) = 0.72e^{-1.2}$$

$$\Pr(S=3) = \frac{1}{3}\left(0.8\Pr(S=2) + 0.8\Pr(S=1)\right) = 0.40533e^{-1.2}$$

Totaling these 4 probabilities you will find that: $\Pr(S \le 3) = 2.92533e^{-1.2} = 0.88109$.

Solution 4.17

In order to calculate from combinatorial reasoning, we must first filter out the zero terms in the sum and then adjust the frequency distribution (see Section 3.7). Let's set notation:

$$\begin{aligned} S &= Y_1 + \cdots + Y_{N_L} \quad \text{where } N_L \sim \text{Poisson } \lambda = 4 \\ &= Z_1 + \cdots + Z_{N_P} \quad \text{where } Z = Y \mid Y > 0 \text{ and } N_P \sim \text{Poisson } \lambda^* = \lambda \Pr(Y > 0) \end{aligned}$$

From Solution 4.16, we have:

$$\Pr(Z = 1) = \frac{\Pr(Y = 1)}{\Pr(Y > 0)} = \frac{2}{3} \quad , \quad \Pr(Z = 2) = \frac{\Pr(Y = 2)}{\Pr(Y > 0)} = \frac{1}{3} \quad , \quad \lambda^* = 4 \times 0.3 = 1.2$$

Now it is easy to duplicate the probability calculations in Solution 4.16. For example, we have:

$$\Pr(S = 0) = \Pr(N_P = 0) = e^{-\lambda^*} = e^{-1.2}$$

$$\Pr(S = 1) = \Pr(N_P = 1 \text{ and } Z = 1) = e^{-1.2}\,\frac{1.2^1}{1!} \times \frac{2}{3} = 0.8e^{-1.2}$$

It is left to the reader to verify the calculations for $\Pr(S = 2)$ and $\Pr(S = 3)$.

Solution 4.18

$$\begin{aligned} E\left[(S - 3.8)_+\right] &= E[S] - E[S \wedge 3.8] \\ &= \lambda E\left[(X-1)_+\right] - \left(0 \cdot \Pr(S=0) + 1 \cdot \Pr(S=1) + 2 \cdot \Pr(S=2) + 3 \cdot \Pr(S=3) + 3.8 \times \Pr(S \geq 4)\right) \\ &= 4 \times 0.4 - \left(0.8e^{-1.2} + 2 \times 0.72e^{-1.2} + 3 \times 0.40533e^{-1.2} + 3.8 \times 0.11891\right) = 0.10723 \end{aligned}$$

Solution 4.19

Aggregate annual claims follow a compound Poisson distribution:

$$S = (X_1 - 100)_+ + \cdots + \left(X_{N_L} - 100\right)_+ \quad \text{where } N_L \sim \text{Poisson } \lambda = 10$$

The severity model X follows an exponential distribution with mean $\theta = 500$. For each loss $Y = (X - 100)_+$ of the insurer, the reinsurer pays the insurer an amount R equal to the excess of Y over 500, if there is an excess:

$$R = (Y - 500)_+ = \left((X - 100)_+ - 500\right)_+ = (X - 600)_+$$

The total payment by the reinsurer to the insurer is thus $S_{re} = R_1 + \cdots + R_{N_L}$. The pure reinsurance premium is the expected value of S_{re}. Since X follows an exponential distribution with mean $\theta = 500$, we know that $X - 600 \mid X > 600$ follows this same exponential distribution. As a result, we have:

$$E[R] = E\left[(X-600)_+\right] = E\left[X - 600 \mid X > 600\right] \Pr(X > 600) = 500 \times e^{-600/500} = 150.59711$$

$$E[S_{re}] = E[N_L]\, E[R] = 10 \times 150.59711 = 1{,}505.97$$

Solution 4.20

Aggregate annual claims follow a compound Poisson distribution:

$$S = (X_1 - 100)_+ + \cdots + (X_{N_L} - 100)_+ \quad \text{where } N_L \sim \text{Poisson } \lambda = 10$$

The severity model X follows an exponential distribution with mean $\theta = 500$. For each loss $Y = (X - 100)_+$ of the insurer, the reinsurer pays the insurer an amount $R = 0.25\,Y$. In this case it is easy to see that the total reinsurance payments are $S_{Re} = 0.25\,S$. So the pure reinsurance premium is:

$$E[S_{Re}] = 0.25\,E[S] = 0.25 \times \underbrace{4{,}093.65377}_{\text{Solution 4.6}} = 1{,}023.41$$

Solution 4.21

The surplus at the end of the first year will be:

$(2+1.5)\times 1.1 - 0 = 3.85$ with probability 0.7
$(2+1.5)\times 1.1 - 4 = -0.15$ with probability 0.2
$(2+1.5)\times 1.1 - 5 = -1.15$ with probability 0.1

In the last two cases, ruin has already occurred in the first year with probability 0.3.

If the surplus at the end of the first year is 3.85, then the surplus at the end of the second year will be:

$(3.85+1.5)\times 1.1 - 0 = 5.885$ with probability 0.7
$(3.85+1.5)\times 1.1 - 4 = 1.885$ with probability 0.2
$(3.85+1.5)\times 1.1 - 5 = 0.885$ with probability 0.1

In none of these cases has ruin occurred.

Continuing in this way, we find that if the surplus at the end of Year 2 is 5.885, the possible surpluses at the end of Year 3 are 8.1235, 4.1235 and 3.1235. If the surplus at the end of Year 2 is 1.885, the possible surpluses at the end of Year 3 are 3.7235, –0.2765 and –1.2765. If the surplus at the end of Year 2 is 0.885, the possible surpluses at the end of Year 3 are 2.6235, –1.3765 and –2.3765.

Adding up the possibilities, we find that the probability of ruin is:

$$0.3 + (0.7 \times 0.2 \times 0.3) + (0.7 \times 0.1 \times 0.3) = 0.363$$

