



Construction of Actuarial Models

Fourth Edition

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Solutions to practice questions – Chapter 8

Solution 8.1

In the expression $Z\bar{X}_n + (1-Z)\mu_{\text{collateral}}$ only the sample mean \bar{X}_n is random. By the CLT it is approximately normal in distribution, and:

$$E[Z\bar{X}_n + (1-Z)\mu_{\text{collateral}}] = Z\mu + (1-Z)\mu_{\text{collateral}} \quad \text{where } \mu = E[X]$$

$$\text{var}(Z\bar{X}_n + (1-Z)\mu_{\text{collateral}}) = Z^2 \text{var}(\bar{X}_n) = Z^2 \times \frac{\text{var}(X)}{n}$$

Solution 8.2

$$0.98 = \Pr(|\bar{X}_n - \mu| \leq 0.01\mu)$$

$$= \Pr(0.99\mu \leq \bar{X}_n \leq 1.01\mu)$$

Solution 8.3

$$\alpha = 0.10 \Rightarrow z_{\alpha/2} = 1.645$$

$$\alpha = 0.05 \Rightarrow z_{\alpha/2} = 1.960$$

$$\Rightarrow \frac{n_{\text{full}} \text{ with } \alpha=0.05}{n_{\text{full}} \text{ with } \alpha=0.10} = \left(\frac{1.960}{1.645}\right)^2 \times CV^2(X) \bigg/ \left(\frac{1.645}{1.645}\right)^2 \times CV^2(X)$$

$$= \left(\frac{1.960}{1.645}\right)^2 = 1.41965$$

(a 41.97% increase)

Solution 8.4

$$\frac{n_{\text{full with } k=0.025}}{n_{\text{full with } k=0.050}} = \left(\frac{z_{\alpha/2}}{0.025}\right)^2 \times CV^2(X) \bigg/ \left(\frac{z_{\alpha/2}}{0.050}\right)^2 \times CV^2(X)$$

$$= \left(\frac{0.050}{0.025}\right)^2 = 4$$

(an increase of 300%)

Solution 8.5

$$\frac{n_{\text{full with } 1.1CV_X}}{n_{\text{full with } CV_X}} = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times (1.1CV(X))^2 \bigg/ \left(\frac{z_{\alpha/2}}{k}\right)^2 \times CV^2(X)$$

$$= 1.1^2 = 1.21$$

(a 21% increase)

Solution 8.6

$$\text{exponential} \Rightarrow E[X]=\theta, \text{var}(X)=\theta^2 \Rightarrow CV(X)=\frac{\theta^2}{\theta^2}=1$$

$$\alpha=0.10 \Rightarrow z_{\alpha/2}=1.645$$

$$200 = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times (CV(X))^2 = \frac{1.645^2}{k^2}$$

$$\Rightarrow k = \frac{1.645}{\sqrt{200}} = 0.11632$$

Solution 8.7

There is no change in the amount of data required for full credibility. When a random variable is multiplied by a positive constant, then both the mean and standard deviation are also multiplied by this same constant. So the coefficient of variation is unchanged.

Solution 8.8

$$100 = \text{credibility estimate} = Z\bar{X}_3 + (1-Z)\mu_{\text{collateral}}$$

$$= Z \times \frac{100 + 120 + 110}{3} + (1-Z) \times 90$$

$$\Rightarrow Z = 0.50$$

Solution 8.9

$$\sum_{i=1}^{100} x_i = 35 \quad \text{and} \quad \sum_{i=1}^{100} x_i^2 = 48$$

$$\Rightarrow \bar{x}_n = \frac{35}{100} = 0.35$$

$$s_X^2 = \frac{1}{99} \times \left(\sum_{i=1}^{100} x_i^2 - 100 \bar{x}_{100}^2 \right) = \frac{35.75}{99} = 0.36111$$

$$\alpha = 0.20, \quad k = 0.10 \Rightarrow z_{\alpha/2} = z_{0.10} = 1.282$$

$$\Rightarrow n_{\text{full}} \approx \left(\frac{z_{\alpha/2}}{k} \right)^2 \times \frac{s_X^2}{(\bar{x}_n)^2} = \left(\frac{1.282}{0.10} \right)^2 \times \frac{0.36111}{0.35^2} = 484.49 \nearrow 485$$

Solution 8.10

Using the value of n_{full} from Solution 8.9, we have:

$$Z_{\text{partial}} = \sqrt{\frac{100}{485}} = 0.454$$

Solution 8.11

Using the value of n_{full} from Solution 8.9 and the value of Z_{partial} from Solution 8.10, we have:

$$Z \bar{X}_n + (1-Z) \mu_{\text{collateral}} = 0.454 \times \frac{35}{100} + 0.556 \times 0.28 = 0.312$$

Solution 8.12

$$E[X] = \theta, \quad \text{var}(X) = \theta^2 \Rightarrow CV(X) = \frac{\text{var}(X)}{(E[X])^2} = 1$$

$$k = 0.10, \quad \alpha = 0.10 \Rightarrow z_{\alpha/2} = 1.645 \Rightarrow$$

$$n_{\text{full}} = \left(\frac{z_{\alpha/2}}{k} \right)^2 \times CV(X) = \left(\frac{1.645}{0.10} \right)^2 = 270.6 \nearrow 271$$

Solution 8.13

$$\bar{x}_{100} = \frac{\sum_{i=1}^{100} x_i}{100} = \frac{5,000}{100} = 50$$

$$s_X^2 = \frac{\sum_{i=1}^{100} x_i^2 - 100\bar{x}_{100}^2}{100-1} = \frac{2,000,000 - 100 \times 50^2}{99} = \frac{1,750,000}{99} = 17,676.77$$

$$k = 0.05, \alpha = 0.05 \Rightarrow z_{\alpha/2} = 1.96 \Rightarrow$$

$$n_{\text{full}} \approx \left(\frac{z_{\alpha/2}}{k} \right)^2 \times \frac{s_X^2}{(\bar{x}_n)^2} = \left(\frac{1.96}{0.05} \right)^2 \times \frac{17,676.77}{50^2} = 10,865.1 \nearrow 10,866$$

Solution 8.14

Here is one example with the negative binomial:

$$n_{\text{full}} = \left(\frac{z_{\alpha/2}}{k} \right)^2 \times CV^2(N) = \left(\frac{z_{\alpha/2}}{k} \right)^2 \times \frac{r\beta(1+\beta)}{(r\beta)^2} = \left(\frac{z_{\alpha/2}}{k} \right)^2 \times \frac{1+\beta}{r\beta} \Rightarrow$$

$$\underbrace{n_{\text{full}} E[N]}_{\text{total number of claims}} = \left(\frac{z_{\alpha/2}}{k} \right)^2 \times \frac{1+\beta}{r\beta} \times r\beta = \left(\frac{z_{\alpha/2}}{k} \right)^2 \times (1+\beta)$$

Solution 8.15

We are given:

$$k = 0.05, \alpha = 0.10 \Rightarrow z_{\alpha/2} = 1.645$$

Using the data and moment formulas for the negative binomial distribution together with the method of moments, we have:

$$r\beta = E[N] = \bar{n}_{30} = \frac{\sum_{i=1}^{30} n_i}{30} = \frac{27}{30} = 0.9$$

$$r\beta(1+\beta) = \text{var}(N) = \frac{\sum_{i=1}^{30} (n_i - \bar{n}_{30})^2}{30} = \frac{\sum_{i=1}^{30} n_i^2 - 30\bar{n}_{30}^2}{30} = \frac{170 - 30 \times 0.9^2}{30} = 4.85667$$

$$\Rightarrow 1+\beta = \frac{\text{var}(N)}{E[N]} = \frac{4.85667}{0.9} \Rightarrow \beta = 4.39630$$

$$\Rightarrow r = \frac{0.9}{\beta} = 0.20472$$

So the number of years of claim counts required for full credibility is:

$$n_{\text{full}} = \left(\frac{z_{\alpha/2}}{k} \right)^2 \times \frac{r\beta(1+\beta)}{(r\beta)^2} = \left(\frac{1.645}{0.05} \right)^2 \times \frac{1+\beta}{r\beta} \approx 1,082.41 \times \frac{5.39630}{0.9} = 6,490.01 \nearrow 6,491$$

Solution 8.16

Using the results of Solution 8.15, we have:

$$n_{\text{full}} \times E[N] \approx n_{\text{full}} \times r\beta = 6,491 \times 0.9 = 5,841.9 \nearrow 5,842$$

Solution 8.17

For a 2-parameter Pareto distribution, the squared coefficient of variation is:

$$CV^2(X) = \frac{\sigma_X^2}{(E[X])^2} = \left(\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)} \right) / \left(\frac{\theta}{\alpha-1} \right)^2 = \frac{\alpha}{\alpha-2}$$

From the data we can estimate the Pareto parameter α by the method of moments:

$$\begin{aligned} \frac{\theta}{\alpha-1} &= E[X] = \sum_{i=1}^{100} x_i / 100 = 9,495 / 100 = 94.95 \\ \frac{2\theta^2}{(\alpha-1)(\alpha-2)} &= E[X^2] = \sum_{i=1}^{100} x_i^2 / 100 = 5,314,750 / 100 = 53,147.5 \\ \Rightarrow \frac{2(\alpha-1)}{\alpha-2} &= \frac{E[X^2]}{(E[X])^2} = \frac{53,147.5}{94.95^2} = 5.89512 \\ \Rightarrow \alpha &= 2.51346 \end{aligned}$$

Now we can determine the number of claims required for full credibility:

$$\begin{aligned} k = 0.05, \alpha = 0.10 &\Rightarrow z_{\alpha/2} = 1.645 \Rightarrow \\ n_{\text{full}} &= \left(\frac{z_{\alpha/2}}{k} \right)^2 \times CV^2(X) = \left(\frac{1.645}{0.05} \right)^2 \times \frac{\alpha}{\alpha-2} \approx 1,082.41 \times \frac{\hat{\alpha}}{\hat{\alpha}-2} \\ &= 1,082.41 \times \frac{2.51346}{0.51346} = 5,298.5 \nearrow 5,299 \end{aligned}$$

Solution 8.18

Using the results of Solution 8.17, the number of dollars of claims for full credibility is:

$$n_{\text{full}} \times E[X] \approx 5,299 \times \bar{x}_{100} = 503,140.05 \nearrow 503,141$$

Solution 8.19

The number of years of observation total annual claims for full credibility for a compound Poisson model with gamma severity is:

$$\begin{aligned} n_{\text{full}} &= \left(\frac{z_{\alpha/2}}{k} \right)^2 \times \frac{E[N] \text{var}(X) + (E[X])^2 \text{var}(N)}{(E[N]E[X])^2} \quad (\text{general}) \\ &= \left(\frac{z_{\alpha/2}}{k} \right)^2 \times \frac{\lambda E[X^2]}{(\lambda E[X])^2} \quad (\text{compound Poisson}) \\ &= \left(\frac{z_{\alpha/2}}{k} \right)^2 \times \frac{\alpha(\alpha+1)\theta^2}{\lambda(\alpha\theta)^2} = \left(\frac{z_{\alpha/2}}{k} \right)^2 \times \frac{\alpha+1}{\lambda\alpha} \end{aligned}$$

We are given:

$$k = 0.05, \alpha = 0.10 \Rightarrow z_{\alpha/2} = 1.645$$

We are told that the severity model is gamma and that in the past 4 years there have been 432 claims paid with a total claim amount equal to 583,200. Also, the total sum of squared claim payments is 1,180,980,000. From this data and the method of moments we can estimate the unknown Poisson and gamma parameters:

$$\begin{aligned} \lambda = E[N] = \bar{n}_4 &= \frac{432}{4} = 108 \\ \alpha\theta = E[X] = \bar{x}_{432} &= \frac{583,200}{432} = 1,350 \\ \alpha(\alpha+1)\theta^2 = E[X^2] &= \frac{\sum_{i=1}^{432} x_i^2}{432} = \frac{1,180,980,000}{432} = 2,733,750 \\ \Rightarrow \frac{\alpha+1}{\alpha} &= \frac{2,733,750}{1,350^2} = 1.5 \Rightarrow \hat{\alpha} = 2 \end{aligned}$$

Finally, we have:

$$n_{\text{full}} = \left(\frac{z_{\alpha/2}}{k} \right)^2 \times \frac{\alpha+1}{\lambda\alpha} = \left(\frac{1.645}{0.05} \right)^2 \times \frac{2+1}{108 \times 2} = 15.03 \nearrow 16$$

Solution 8.20

Using the results of Solution 8.19, the number of claims needed for full credibility is:

$$n_{\text{full}} \times E[N] \approx 16 \times 108 = 1,728$$