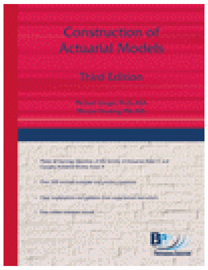

Construction of Actuarial Models

Third Edition

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Solutions to practice questions – Chapter 9

Solution 9.1

In the expression $Z\bar{X}_n + (1-Z)\mu_{\text{collateral}}$ only the sample mean $\bar{X}_n$ is random. By the CLT it is approximately normal in distribution, and:

$$E\left[Z\bar{X}_n + (1-Z)\mu_{\text{collateral}}\right] = Z\mu + (1-Z)\mu_{\text{collateral}} \quad \text{where } \mu = E[X]$$

$$\text{var}\left(Z\bar{X}_n + (1-Z)\mu_{\text{collateral}}\right) = Z^2\,\text{var}\left(\bar{X}_n\right) = Z^2 \times \frac{\text{var}(X)}{n}$$

Solution 9.2

$$\begin{aligned} 0.98 &= \Pr\left(\left|\bar{X}_n - \mu\right| \le 0.01\mu\right) \\ &= \Pr\left(0.99\mu \le \bar{X}_n \le 1.01\mu\right) \end{aligned}$$

Solution 9.3

$$\alpha = 0.10 \;\Rightarrow\; z_{\alpha/2} = 1.645$$

$$\alpha = 0.05 \;\Rightarrow\; z_{\alpha/2} = 1.960$$

$$\Rightarrow \frac{n_{\text{full}} \text{ with } \alpha = 0.05}{n_{\text{full}} \text{ with } \alpha = 0.10} = \left(\frac{1.960}{k}\right)^2 \times CV^2(X) \bigg/ \left(\frac{1.645}{k}\right)^2 \times CV^2(X)$$

$$= \left(\frac{1.960}{1.645}\right)^2 = 1.41965$$

(a 41.97% increase)

Solution 9.4

$$\frac{n_{\text{full}} \text{ with } k = 0.025}{n_{\text{full}} \text{ with } k = 0.050} = \left(\frac{z_{\alpha/2}}{0.025}\right)^2 \times CV^2(X) \bigg/ \left(\frac{z_{\alpha/2}}{0.050}\right)^2 \times CV^2(X)$$

$$= \left(\frac{0.050}{0.025}\right)^2 = 4$$

(an increase of 300%)

Solution 9.5

$$\frac{n_{\text{full}} \text{ with } 1.1CV_X}{n_{\text{full}} \text{ with } CV_X} = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times \left(1.1CV(X)\right)^2 \bigg/ \left(\frac{z_{\alpha/2}}{k}\right)^2 \times CV^2(X)$$

$$= 1.1^2 = 1.21$$

(a 21% increase)

Solution 9.6

$$\text{exponential} \Rightarrow E[X] = \theta \text{ , var}(X) = \theta^2 \Rightarrow CV(X) = \frac{\theta^2}{\theta^2} = 1$$

$$\alpha = 0.10 \Rightarrow z_{\alpha/2} = 1.645$$

$$200 = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times \left(CV(X)\right)^2 = \frac{1.645^2}{k^2}$$

$$\Rightarrow k = \frac{1.645}{\sqrt{200}} = 0.11632$$

Solution 9.7

There is no change in the amount of data required for full credibility. When a random variable is multiplied by a positive constant, then both the mean and standard deviation are also multiplied by this same constant. So the coefficient of variation is unchanged.

Solution 9.8

$$100 = \text{credibility estimate} = Z\bar{X}_3 + (1-Z)\mu_{\text{collateral}}$$

$$= Z \times \frac{100 + 120 + 110}{3} + (1-Z) \times 90$$

$$\Rightarrow Z = 0.50$$

Solution 9.9

$$\sum_{1=1}^{100} x_i = 35 \quad \text{and} \quad \sum_{1=1}^{100} x_i^2 = 48$$

$$\Rightarrow \quad \bar{x}_n = \frac{35}{100} = 0.35$$

$$s_X^2 = \frac{1}{99} \times \left(\sum_{i=1}^{100} x_i^2 - 100\,\bar{x}_{100}^2 \right) = \frac{35.75}{99} = 0.36111$$

$$\alpha = 0.20 \quad , \quad k = 0.10 \Rightarrow z_{\alpha/2} = z_{0.10} = 1.282$$

$$\Rightarrow \quad n_{\text{full}} \approx \left(\frac{z_{\alpha/2}}{k} \right)^2 \times \frac{s_X^2}{(\bar{x}_n)^2} = \left(\frac{1.282}{0.10} \right)^2 \times \frac{0.36111}{0.35^2} = 484.49 \nearrow 485$$

Solution 9.10

Using the value of n_{full} from Solution 9.9, we have:

$$Z_{\text{partial}} = \sqrt{\frac{100}{485}} = 0.454$$

Solution 9.11

Using the value of n_{full} from Solution 9.9 and the value of Z_{partial} from Solution 9.10, we have:

$$Z\,\bar{X}_n + (1 - Z)\,\mu_{\text{collateral}} = 0.454 \times \frac{35}{100} + 0.556 \times 0.28 = 0.312$$

Solution 9.12

$$E[X] = \theta \ , \ \text{var}(X) = \theta^2 \Rightarrow CV(X) = \frac{\text{var}(X)}{\left(E[X]\right)^2} = 1$$

$$k = 0.10 \ , \ \alpha = 0.10 \Rightarrow z_{\alpha/2} = 1.645 \quad \Rightarrow$$

$$n_{\text{full}} = \left(\frac{z_{\alpha/2}}{k} \right)^2 \times CV(X) = \left(\frac{1.645}{0.10} \right)^2 = 270.6 \nearrow 271$$

Solution 9.13

$$\bar{x}_{100} = \sum_{i=1}^{100} x_i \Big/ 100 = \frac{5,000}{100} = 50$$

$$s_X^2 = \frac{\sum_{i=1}^{100} x_i^2 - 100\bar{x}_{100}^2}{100-1} = \frac{2,000,000 - 100 \times 50^2}{99} = \frac{1,750,000}{99} = 17,676.77$$

$$k = 0.05 \ , \ \alpha = 0.05 \Rightarrow z_{\alpha/2} = 1.96 \Rightarrow$$

$$n_{\text{full}} \approx \left(\frac{z_{\alpha/2}}{k}\right)^2 \times \frac{s_X^2}{(\bar{x}_n)^2} = \left(\frac{1.96}{0.05}\right)^2 \times \frac{17,676.77}{50^2} = 10,865.1 \nearrow 10,866$$

Solution 9.14

Here is one example with the negative binomial:

$$n_{\text{full}} = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times CV^2(N) = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times \frac{r\beta(1+\beta)}{(r\beta)^2} = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times \frac{1+\beta}{r\beta} \Rightarrow$$

$$\underbrace{n_{\text{full}}\, E[N]}_{\text{total number of claims}} = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times \frac{1+\beta}{r\beta} \times r\beta = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times (1+\beta)$$

Solution 9.15

We are given:

$$k = 0.05 \ , \ \alpha = 0.10 \Rightarrow z_{\alpha/2} = 1.645$$

Using the data and moment formulas for the negative binomial distribution together with the method of moments, we have:

$$r\beta = E[N] = \bar{n}_{30} = \frac{\sum_{i=1}^{30} n_i}{30} = \frac{27}{30} = 0.9$$

$$r\beta(1+\beta) = \text{var}(N) = \frac{\sum_{i=1}^{30}(n_i - \bar{n}_{30})^2}{30} = \frac{\sum_{i=1}^{30} n_i^2 - 30\bar{n}_{30}^2}{30} = \frac{170 - 30 \times 0.9^2}{30} = 4.85667$$

$$\Rightarrow 1+\beta = \frac{\text{var}(N)}{E[N]} = \frac{4.85667}{0.9} \Rightarrow \beta = 4.39630$$

$$\Rightarrow r = \frac{0.9}{\beta} = 0.20472$$

So the number of years of claim counts required for full credibility is:

$$n_{\text{full}} = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times \frac{r\beta(1+\beta)}{(r\beta)^2} = \left(\frac{1.645}{0.05}\right)^2 \times \frac{1+\beta}{r\beta} \approx 1{,}082.41 \times \frac{5.39630}{0.9} = 6{,}490.01 \nearrow 6{,}491$$

Solution 9.16

Using the results of Solution 9.15, we have:

$$n_{\text{full}} \times E[N] \approx n_{\text{full}} \times r\beta = 6{,}491 \times 0.9 = 5{,}841.9 \nearrow 5{,}842$$

Solution 9.17

For a 2-parameter Pareto distribution, the squared coefficient of variation is:

$$CV^2(X) = \frac{\sigma_X^2}{(E[X])^2} = \left(\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}\right) \Big/ \left(\frac{\theta}{\alpha-1}\right)^2 = \frac{\alpha}{\alpha-2}$$

From the data we can estimate the Pareto parameter α by the method of moments:

$$\frac{\theta}{\alpha-1} = E[X] = \sum_{i=1}^{100} x_i / 100 = 9{,}495/100 = 94.95$$

$$\frac{2\theta^2}{(\alpha-1)(\alpha-2)} = E\left[X^2\right] = \sum_{i=1}^{100} x_i^2 / 100 = 5{,}314{,}750/100 = 53{,}147.5$$

$$\Rightarrow \frac{2(\alpha-1)}{\alpha-2} = \frac{E\left[X^2\right]}{(E[X])^2} = \frac{53{,}147.5}{94.95^2} = 5.89512$$

$$\Rightarrow \alpha = 2.51346$$

Now we can determine the number of claims required for full credibility:

$$k = 0.05,\ \alpha = 0.10 \Rightarrow z_{\alpha/2} = 1.645 \Rightarrow$$

$$n_{\text{full}} = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times CV^2(X) = \left(\frac{1.645}{0.05}\right)^2 \times \frac{\alpha}{\alpha-2} \approx 1{,}082.41 \times \frac{\hat{\alpha}}{\hat{\alpha}-2}$$

$$= 1{,}082.41 \times \frac{2.51346}{0.51346} = 5{,}298.5 \nearrow 5{,}299$$

Solution 9.18

Using the results of Solution 9.17, the number of dollars of claims for full credibility is:

$$n_{\text{full}} \times E[X] \approx 5{,}299 \times \bar{x}_{100} = 503{,}140.05 \nearrow 503{,}141$$

Solution 9.19

The number of years of observation total annual claims for full credibility for a compound Poisson model with gamma severity is:

$$n_{\text{full}} = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times \frac{E[N]\text{var}(X) + (E[X])^2 \text{var}(N)}{(E[N]E[X])^2} \quad \text{(general)}$$

$$= \left(\frac{z_{\alpha/2}}{k}\right)^2 \times \frac{\lambda E\left[X^2\right]}{(\lambda E[X])^2} \quad \text{(compound Poisson)}$$

$$= \left(\frac{z_{\alpha/2}}{k}\right)^2 \times \frac{\alpha(\alpha+1)\theta^2}{\lambda(\alpha\theta)^2} = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times \frac{\alpha+1}{\lambda\alpha}$$

We are given:

$$k = 0.05 \ , \ \alpha = 0.10 \ \Rightarrow \ z_{\alpha/2} = 1.645$$

We are told that the severity model is gamma and that in the past 4 years there have been 432 claims paid with a total claim amount equal to 583,200. Also, the total sum of squared claim payments is $1,180,980,000$. From this data and the method of moments we can estimate the unknown Poisson and gamma parameters:

$$\lambda = E[N] = \bar{n}_4 = \frac{432}{4} = 108$$

$$\alpha\theta = E[X] = \bar{x}_{432} = \frac{583,200}{432} = 1,350$$

$$\alpha(\alpha+1)\theta^2 = E\left[X^2\right] = \sum_{i=1}^{432} x_i^2 / 432 = \frac{1,180,980,000}{432} = 2,733,750$$

$$\Rightarrow \frac{\alpha+1}{\alpha} = \frac{2,733,750}{1,350^2} = 1.5 \ \Rightarrow \ \hat{\alpha} = 2$$

Finally, we have:

$$n_{\text{full}} = \left(\frac{z_{\alpha/2}}{k}\right)^2 \times \frac{\alpha+1}{\lambda\alpha} = \left(\frac{1.645}{0.05}\right)^2 \times \frac{2+1}{108 \times 2} = 15.03 \nearrow 16$$

Solution 9.20

Using the results of Solution 9.19, the number of claims needed for full credibility is:

$$n_{\text{full}} \times E[N] \approx 16 \times 108 = 1,728$$