Financial Mathematics: Derivatives Markets

Course notes for SOA Exam FM

- Meets learning objectives of the Society of Actuaries Exam FM
- Over 50 worked examples and practice questions
- Clear explanations and guidance from experienced instructors
- Free online solutions manual

BPP
PROFESSIONAL
EDUCATION

Introduction

These Exam FM notes are designed to be a replacement for the material on the Society of Actuaries syllabus from *Derivatives Markets.* Each study session corresponds to the study sessions in our Study Program Guide and includes practice questions at the end of each session. The solutions for these practice questions can be found online at *www.bpptraining.com*.

BPP's Question & Answer bank (available separately) is divided into the corresponding study sessions. You'll find full solutions to every question at the end of the Question & Answer Bank.

Good luck in the exam.

BPP Exam FM Tutor Team

November 2015

Study Session 9

Introduction to derivatives

➔ *This session's course notes cover a relatively straightforward introduction to derivatives, but don't worry, it will get much more complicated soon enough!*

➔ *Our policy on rounding is to keep full accuracy within intermediate calculations, even though an intermediate result may be shown as a rounded value.*

9.1 Derivative security overview

A *derivative* security is a financial instrument whose value is derived from the return or price of another asset, which is called the *underlying asset*. In practice, derivatives are based upon a wide range of underlying assets, including:

- financial securities
- commodities, such as crude oil, gold and coffee beans
- indexes, based on security prices and commodity prices.

More recently, derivatives have been developed that derive their value from an underlying liability or even an underlying economic index, such as employment statistics. In fact, in principle, a derivative can be agreed based on almost any underlying asset.

In the study sessions that follow, we will consider such derivatives as forwards, futures, options, and swaps, as well as various combinations of these derivatives.

How derivatives are used

In general, derivatives are used to manage exposure to risk and returns, by a wide range of investors and also by the producers of commodities. For example, the producer of a product is exposed to the risk that the price of the product falls before they are able to bring it to market. As we shall see in Study Session 12, this risk could be reduced using a forward contract.

Likewise, investors are exposed to many risks, including the risk that asset prices will change over time. The potential for an asset's price to rise and fall is sometimes called *price volatility risk* and can be measured by the standard deviation of an asset's price over time.

While investors may be content to be exposed to price volatility risk, they often prefer to manage risk according to their own unique characteristics. Derivatives are a tool that can be used in the *risk-management* process either to increase risk or mitigate risk depending upon the investor's preference. This theme of risk management appears throughout the derivatives material.

As far as we are concerned in this course, there are five primary uses of derivatives:

1. Derivatives can be used for *speculation* to increase exposure to risk and hence investment returns. We'll discuss this in Study Session 11. Derivatives also enable the potential gain (or loss) from speculation to be *leveraged* (*ie* magnified) relative to the initial investment.
2. Derivatives can be used as a *hedge* to reduce exposure to risk. We'll discuss this in detail in Study Sessions 12, 13 and 14.
3. Derivatives can be used to *reduce transaction costs*, such as commissions and other trading costs, compared to trading in the underlying assets themselves.
4. Derivatives can be mis-priced with regard to the underlying assets. Investors may therefore be able to use derivatives to exploit such pricing anomalies and make *arbitrage profits*. We'll discuss this in Study Session 13.
5. Finally, derivatives can be used to take advantage of differences in how tax laws, accounting rules, and other regulations may apply differently to derivatives compared to the underlying assets. Investors may therefore use derivatives to take advantage of this fact using a process called *regulatory arbitrage*. We'll see an example of this in Chapter 12.

More generally, by offering an alternative to simply buying or selling the underlying asset itself, derivatives increase the range of possibilities to investors. In particular, derivatives can be used to create financial products that generate an investor's required set of payoffs in order to satisfy that investor's desired risk exposure. This process is called *financial engineering*.

Derivative perspectives

In addition to considering their uses, we can also consider derivatives from three different user perspectives:

1. the *end-user*, typically an investor (*eg* individuals, insurance companies, mutual funds, hedge funds), but sometimes the producer of a product, employs a derivative in the ways described above, *eg* to gain a desired risk exposure.
2. the *market-maker*, through whom end-users trade derivatives, and who therefore serves as an intermediary between different end users. Market-makers charge a fee for this service, which we'll consider in more detail soon. The market-maker may want to take an offsetting position (*ie* a *hedged position*) to manage his or her own risk exposure.
3. the *economic observer*, such as a regulator or an economist, who observes derivative transactions, how derivatives are used, the operation of derivative markets *etc* and may even set rules for the transaction.

9.2 Financial markets

Derivatives play an increasingly important role in financial markets. Financial markets connect those who are willing to assume a certain type of risk with those who are willing to sell it. More specifically, financial markets:

- *facilitate transactions* between investors, who wish buy and sell exposure to different risks
- allow companies to *raise capital*, via the issuance of stocks and bonds
- allow companies to use derivatives to *hedge or insure* against risk exposure
- enable investors to *lower transactions costs* (*eg* with low-cost index mutual funds)
- permit the *diversification* of risks, to reduce exposure to price volatility risk
- allow *risk pooling* with others willing to share those risks, which enables insurance costs to be reduced
- *increase efficiency*, by allowing undesired risks to be sold to parties who are more willing to bear them.

The role of risk sharing is very important in the context of insurance. Insurance companies assume risk, pooling the risks from a significant number of policyholders. Insurance companies can hedge this risk with *reinsurance* to cede some of their liabilities to a reinsurance company willing to bear it. A reinsurance company can in turn share some of its risk with other parties, *eg* by selling a *catastrophe bond* that does not need to be repaid in full in the event of a specified catastrophe, such as a hurricane.

Diversifiable vs. nondiversifiable risk

Portfolio theory divides risk into two categories, namely diversifiable and nondiversifiable.

Diversifiable risk is company-specific risk that can be eliminated by risk sharing via diversification within a financial market.

Nondiversifiable risk is exposure to a market risk that cannot be diversified away.

Financial markets enable investors to eliminate their exposure to diversifiable risk, by sharing that risk with many others. At the same time, they also enable nondiversifiable risk to be passed on to those most willing to bear it.

9.3 *The practicalities of using derivatives*

The use of derivatives by companies and investors to manage their risks has increased hugely over recent years, although data to measure the extent of derivative usage have either not been very precise or readily available.

Derivatives may be traded on an *exchange* or they may be traded *over-the-counter*. As the name implies, exchanged-traded derivatives are traded on an exchange, such as the Chicago Board of Trade (CBOT). Exchanges standardize derivatives in order to facilitate trading and are highly regulated. Such derivatives are therefore quick, cheap and easy to trade. Over-the-counter (OTC) derivatives are not traded on an exchange, but are bought and sold through dealers and investment banks. OTC derivatives must be individually negotiated, with the advantage that they can be customized to meet the needs of a particular investor. The OTC market is not as regulated as the exchange market and offers more confidentiality among participants.

9.4 *Buying and short selling financial assets*

Buying and selling both an asset and a derivative involves *transaction costs*. There are two main types of transaction costs: the broker's commission and the bid-ask spread.

The *commission* is simply a transaction fee, which can be expressed as either a percentage of the price or as a flat fee, *eg* as 0.1% of the total price paid or as $20 per transaction.

The *bid-ask spread* is the difference between the *bid price*, at which the broker will buy the asset, and the higher *ask price*, at which the broker sells an asset. Note that these terms are named from the broker's perspective. In other words, the bid price is the buying price (*ie* the bidding price) the broker pays for a share of stock, which is the selling price the investor receives. The ask price is also known as the *offer price*, which is the price the broker offers to sell the share of stock, so the ask price is the buying price the investor pays. As such, the bid price is less than the ask (offer) price, and the broker profits from the bid-ask spread.

Buying an asset

When an investor buys an asset, it is referred to as a *long position* in the asset, and the investor usually has an opinion that the asset price will increase. The buyer therefore hopes to profit from selling the asset later at a higher price and is said to be *bullish* on the asset's future prospects. The old adage, "buy low and sell high" applies, but to be more precise, let's say "buy low now and sell high later" so that we can compare a long position with a short position. Likewise, an investor or producer who already owns an asset or product is also said to have a long position in the asset and will again profit from a price rise.

Finally, note that buying an asset is like *lending cash* since the buyer pays cash today to buy the asset and receives cash in the future when the asset is ultimately sold.

Short selling an asset

A *short sale* of an asset involves the sale of an asset now that is not currently owned by the seller and the later repurchase and return of the asset to its original owner.

A short sale is referred to as a *short position* in the asset, and the short seller usually believes that the price of the asset will fall. The short seller may be *speculating* by hoping to profit by buying back the asset in the future at a lower price than the short seller initially sold it at. The short seller is said to be *bearish* on the asset's future prospects. A short sale can be described as "sell high now, buy back low later."

Example

For example, suppose that shares in XYZ com are priced at \$10 each and Julie thinks that they are likely to fall in price in the near future. Julie could then arrange to short sell 1,000 shares at \$10, *ie* a total of \$10,000. If she is proved correct and the shares fall in price to \$9 each, then she could buy them back for \$9,000, thereby netting a profit of \$1,000, excluding costs.

In fact, a short sale involves borrowing the asset now, since it is not currently owned by the seller. This is usually done through a broker who is holding the asset for another client. The broker is willing to lend the asset to the short seller since the short seller promises to buy the asset back later and return the asset to the broker, which essentially returns the asset to its original owner. In addition, the short seller will normally pay a commission to the broker.

One potential complication with short sales is when the asset pays a dividend during the short sale period. The original owner of the asset is entitled to receive this dividend, but the asset is not in the owner's brokerage account since the short seller has borrowed the asset and sold it. So the short seller must pay any dividend during the short sale period to the original owner of the asset to keep the original owner's position whole.

Short selling an asset is like *borrowing cash*, a form of financing, since the short seller receives cash now and must pay at least some of it back later (and possibly more), depending upon the future price of the asset.

The lease rate of an asset

The *lease rate* of an asset refers to the payment made by the borrower of an asset to the lender of the asset.

So, the dividend payment from the short seller to the original owner of a share is an example of the *lease rate* of the asset, here the share. In fact, any payment from the borrower (short seller) to the lender (broker or original owner of the asset) is a lease rate payment of the asset.

Credit risk in short selling

With short selling, there is always a risk that the short seller may be unable to buy back the asset at a later date and return it to the original owner, especially if the asset price rises during the short sale period. This risk is called *credit risk*, since the short seller is a creditor of the original owner, and will be of concern to the broker who agreed to the short sale.

To mitigate this risk, the broker requires the short seller to deposit *collateral* to help cover any future losses. This collateral is called the *margin* and it is deposited into a margin account with the broker. The higher the margin, the greater the protection it offers against the credit risk that may arise should the asset price rise during the short sale period. If the margin is greater than the price at which the asset is sold by the short seller, the excess amount over the sale price is called the *haircut*. The margin requirement is often expressed as a percentage of the sale price.

Example

For example, suppose that when Julie short sold the 1,000 XYZ com shares worth $10,000, she was required to deposit margin of 110% of the sale price. This would have corresponded to a margin deposit of $11,000 and a haircut of $1,000.

Scarcity in short selling

Since the short seller has deposited collateral in the form of margin with the broker, the short seller may expect the broker to pay interest on the margin during the short sale period. The margin interest rate that the broker pays the short seller is called the *repo rate* in the bond market and the *short rebate* in the stock market.

If the asset is scarce or in high demand, the broker may only be willing to pay a low interest rate. If the asset is not scarce, the broker may offer a higher interest rate. The difference between the market interest rate and the margin interest rate is essentially a cost to the short seller and additional compensation to the broker.

Example

So, for example, if the short rebate on Julie's margin deposit is 4%, whereas the market interest rate is 5%, then the 1% difference represents an additional cost to Julie and a fee to the broker.

Short interest

Finally, the short interest measure provides an indication of how other investors view the future prospects of an asset.

Short interest can be expressed either as a number or a percentage. As a number, short interest is the absolute number of shares of a stock that has been sold short and not yet closed. Alternatively, as a percentage, short interest is the total number of shorted shares divided by the total number of shares issued by the underlying company.

Recall that a stock may be sold short if the investor thinks the stock price will go down. Consequently, if the short interest measure decreases significantly, it is an indication that investors are becoming less bearish on the stock, since the number of shorted shares has declined. If the short interest measure increases significantly, it is an indication that investors are becoming more bearish on the stock, since the number of shorted shares has increased.

Short sale summary

The short seller borrows a number of shares from a broker and deposits collateral in a margin account with the broker. The short seller sells the borrowed shares at the initial market price of the shares, which the short seller hopes will be higher than the future price of the shares. After a period of time, the short seller *closes* the short position by buying the shares back at the future price of the shares. If the future price of the shares is less than the initial price, the short seller has earned a profit (excluding costs), based on the fall in the share price. Otherwise, the short seller has incurred a loss. After the shares are repurchased, the short seller returns the shares to the broker and the short position has been *covered*. The money in the margin account is then returned to the short-seller.

➔ *Short sales are also covered in Chapter 6 of BPP's text 'Financial Mathematics'.*

Study Session 9 –Practice Questions

Question 9.1

You short sell 500 shares of a stock whose bid and ask prices are $50.24 and $50.48 respectively.

What is the dollar spread on 500 shares?

Question 9.2

Ninety days after the transaction described in Question 9.1 you cover the short position. At this time the bid and ask prices are $48.12 and $48.36 respectively.

What profit have you made after 90 days?

Question 9.3

Commission of 0.25% is paid on the selling and closing transactions in Questions 9.1 and 9.2.

How is your profit affected by the payment of these commissions?

Question 9.4

ABC com has one million shares in issue, 20,000 of which are owned by Investor A. Investor B borrows 5,000 shares from Investor A and short sells them to Investor C. At the same time, Investor D borrows 10,000 shares from Investor A and sells them to Investor E. A week later, Investor B buys 2,500 ABC shares and returns them to Investor A.

Determine the resulting short interest in ABC shares, assuming there has been no other short selling.

Question 9.5

Which of the following statements about the short sale of a bond are true?

I. Short selling is like borrowing cash.

II. The short seller is bullish on the bond.

III. Coupon payments to the original owner of the bond represent the lease rate of the bond.

(A) I only

(B) I and II only

(C) I and III only

(D) I, II and III

(E) The correct answer is not given by (A), (B), (C) or (D).

11.2 Buying insurance

Floor

A *floor* is the combination of a *long position in the underlying asset and a long position in a put*.

When an investor owns an asset, and hence has a long position in that asset, the investor is exposed to downside price risk. A long put pays a positive amount when the underlying asset's price goes down, so a long put provides insurance for the long asset. That is, the put provides a floor for the value of the asset, while at the same time allowing the investor to participate in any price gains above the strike price.

Example

An investor owns (*ie* is long in) a share in the stock of X&Y com. The current stock price is \$50, which the investor is worried might fall. He therefore pays \$3 to buy a 1-year put option on the stock, with a strike price of \$45.

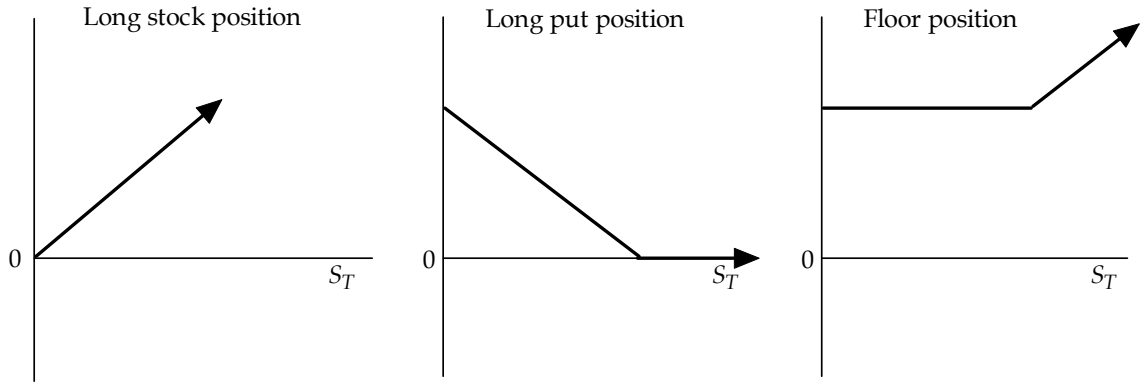
The following table shows how the payoffs from each of the stock, the put and the floor strategy vary with the stock price in one year's time. In particular, it shows how the floor strategy provides a floor for the value of the underlying asset.

Stock price at $T = 1$	Stock payoff	Put payoff	Floor payoff
30	30	15	45
40	40	5	45
50	50	0	50
60	60	0	60

The floor position is also sometimes called an *insured asset*. If the put is purchased when the investor already owns the underlying asset, the position is known as a *protective put*, whereas if it is purchased at the same time as the underlying asset, the position is known as a *married put*.

Floor – payoffs

The payoffs at expiration of a long asset (*eg* stock), a long put, and the combined position (the floor) are shown below.



Prepaid forward price

The prepaid forward price, $F_{0,T}^{P}$, paid at time 0 for an asset to be delivered at time T depends on whether the asset pays dividends and whether those dividends are discrete or continuous:

- No Dividends: $F_{0,T}^{P} = S_0$
- Discrete dividends: $F_{0,T}^{P} = S_0 - PV_{0,T}(Div)$
- Continuous dividends: $F_{0,T}^{P} = S_0 e^{-\delta T}$

In all three cases, the prepaid forward price is just the present value of the forward price:

$$F_{0,T}^{P} = F_{0,T} e^{-rT} = PV_{0,T}(F_{0,T})$$

The above formulas for a dividend-paying stock assume that we know the future dividends with certainty. They are also based on the assumption that there are no arbitrage opportunities, which is the assumption that underlies all derivative pricing. So, let's consider in more detail what is meant by arbitrage and arbitrage opportunities.

Arbitrage

An *arbitrage opportunity* is a position that allows the investor to earn profits with no net investment and no risk of loss. A more technical definition of arbitrage appears in the box below.

Arbitrage

An *arbitrage opportunity* is an investment position that:

1. does not require a cash injection from the investor, and
2. has a nonzero probability of producing a positive cash flow, and
3. a zero probability of producing a loss.

Arbitrage opportunities are basically free money. Therefore we do not expect to see many arbitrage opportunities in the marketplace, and indeed, a key idea underlying the pricing of derivatives is that arbitrage opportunities should not exist. A *fair price* of an asset is one that does not allow an arbitrage opportunity.

If, within the context of a valid pricing model, we find that an assumption implies that arbitrage is possible, then we must reject that assumption. This technique is also used to draw conclusions about the relationships between the prices of various derivatives, such as put-call parity (see Study Session 11). We will also soon learn to recognize arbitrage opportunities and how to exploit them.

If an observed market price does not equal the fair price, then an arbitrage opportunity exists. An arbitrage opportunity is exploited by buying the cheaper asset and selling the higher priced asset (buy low, sell high).

➔ *A question at the end of this chapter looks at an example of how this might work.*

14.5 The objectives of swapping interest rates

As mentioned earlier, interest rate swaps can be used for:

- speculation
- hedging, including reducing the risk of mismatched assets and liabilities
- arbitrage – in particular, reducing the cost of borrowing based on the principle of comparative advantage.

Here we briefly consider some further issues relating to the uses of swaps.

Speculation

Interest rate swaps may be used to speculate on future interest rates if it is cheaper and more convenient to do so using swaps rather than actual securities.

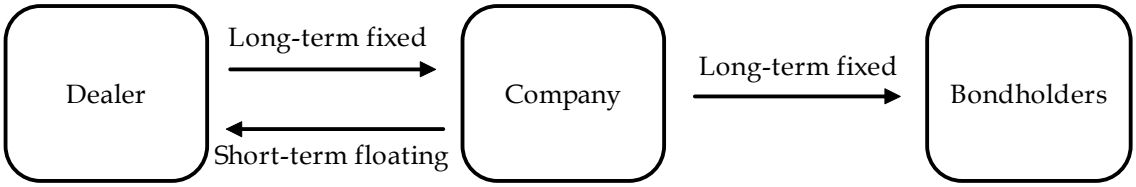
For example, suppose a trader thought that interest rates were likely to decrease. The trader could borrow and buy bonds to lock in the current interest rate, but this would be somewhat costly to do. Instead, the trader could enter into a pay floating (and receive fixed) swap, which would be cheaper and easier to do.

Separating credit risk and interest rate risk

Interest rate swaps also allow a company to separate its credit risk and its interest rate risk, which increases market efficiency since those risks can be individually borne by parties willing to hold one risk but not the other.

Whereas long-term bond holders face both credit risk and interest rate risk, short-term lenders face mostly credit risk – as there is much less time for market conditions to change. So, although a company might wish to borrow short term if it believes that short-term interest rates will be lower, lenders may prefer not to lend large amounts on a short-term basis, for fear of being over-exposed to the credit risk from a particular company. However, the company may be able to use a suitable interest rate swap to swap a long-term interest rate for a (lower) short-term interest rate, so largely separating credit risk and interest rate risk, and enabling itself to benefit from lower short-term interest rates.

The following diagram illustrates how a company could swap a higher long-term interest rate for a lower short-term interest rate:



In this example, the company started by paying a higher-long term interest rate, with its bondholders bearing both credit and interest rate risks. The swap then enabled it to end up paying a lower-short term interest rate to the dealer. This is possible because unlike the bondholders, who still bear most of the credit risk, the dealer is only exposed to default on interest payments, and so faces only a limited credit risk as the swap doesn't involve any exchange of principal.