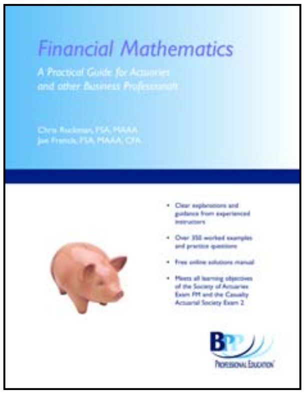

Financial Mathematics

A Practical Guide for Actuaries and other Business Professionals

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Solutions to practice questions – Chapter 3

Solution 3.1

The annual increases are \$50. Since $800 = 300 + 10 \times 50$, there are $10 + 1 = 11$ payments. So the last payment is at time 11 years, since the first payment occurred at time 1 year.

	300	350	400	...	800	payment
0	1	2	3	...	11	time
	50	100	150	...	550	increasing payment
	250	250	250	...	250	level payment

The present value is:

$$300v + 350v^2 + 400v^3 + \cdots + 800v^{11} = 250a_{\overline{11}|4\%} + 50(Ia)_{\overline{11}|4\%}$$

Calculating the required values:

$$a_{\overline{11}|4\%} = \frac{1 - 1.04^{-11}}{0.04} = 8.760476711$$

$$\ddot{a}_{\overline{11}|4\%} = \frac{1 - 1.04^{-11}}{1 - 1.04^{-1}} = 9.11089578$$

$$\Rightarrow \quad (Ia)_{\overline{11}|4\%} = \frac{9.11089578 - 11 \times 1.04^{-11}}{0.04} = 49.1376383$$

So the present value is:

$$250 \times 8.760476711 + 50 \times 49.1376383 = \$4,647.00$$

Solution 3.2

Since we have just calculated the present value at time 0 of these payments in Solution 3.1, we can accumulate the present value for 12 years. The accumulated value at 12 years is:

$$4,647.00(1.04)^{12} = \$7,440.00$$

Alternatively, working from first principles, the accumulated value at 11 years is:

$$300(1+i)^{10} + 350(1+i)^9 + 400(1+i)^8 + \cdots + 800 = 250s_{\overline{11}|4\%} + 50(Is)_{\overline{11}|4\%}$$

Calculating the required values:

$$s_{\overline{11}|4\%} = \frac{1.04^{11} - 1}{0.04} = 13.486351$$
$$\ddot{s}_{\overline{11}|4\%} = (1.04)s_{\overline{11}|4\%} = 14.025805$$
$$\Rightarrow \quad (Is)_{\overline{11}|4\%} = \frac{14.025805 - 11}{0.04} = 75.645137$$

So the accumulated value at 11 years is:

$$250 \times 13.486351 + 50 \times 75.645137 = \$7,153.8447$$

The accumulated value at 12 years is:

$$7,153.8447(1.04) = \$7,440.00$$

Solution 3.3

The annual increases are \$70. Since $1,600 = 900 + 10 \times 70$, there are $10 + 1 = 11$ payments. So there are 11 payments and the last payment is at time 10, since the payments started at time 0.

900	970	1,040	1,110	...	1,600	payment
0	1	2	3	...	10	time
70	140	210	280	...	770	increasing payment
830	830	830	830	...	830	level payment

The present value is:

$$900 + 970v + 1,040v^2 + \cdots + 1,600v^{10} = 830\ddot{a}_{\overline{11}|9\%} + 70(I\ddot{a})_{\overline{11}|9\%}$$

Calculating the required values:

$$\ddot{a}_{\overline{11}|9\%} = \frac{1 - 1.09^{-11}}{1 - 1.09^{-1}} = 7.417657701$$
$$\Rightarrow \quad (I\ddot{a})_{\overline{11}|9\%} = \frac{7.417657701 - 11 \times 1.09^{-11}}{1 - 1.09^{-1}} = 38.20808909$$

So the present value is:

$$830 \times 7.417657701 + 70 \times 38.20808909 = \$8,831.22$$

Solution 3.4

Since we already calculated the present value at time 0 in Solution 3.3, we can accumulate that present value for 11 years. The accumulated value at 11 years is:

$$8{,}831.22(1.09)^{11} = \$22{,}788.31$$

Alternatively, working from first principles, the accumulated value at 11 years is:

$$900(1+i)^{11} + 970(1+i)^{10} + 1{,}040(1+i)^{9} + \cdots + 1{,}600(1+i) = 830\ddot{s}_{\overline{11}|9\%} + 70(I\ddot{s})_{\overline{11}|9\%}$$

Calculating the required values:

$$\ddot{s}_{\overline{11}|9\%} = \frac{1.09^{11} - 1}{1 - 1.09^{-1}} = 19.140720$$

$$\Rightarrow \quad (I\ddot{s})_{\overline{11}|9\%} = \frac{19.140720 - 11}{1 - 1.09^{-1}} = 98.593162$$

So the accumulated value at 11 years is:

$$830 \times 19.140720 + 70 \times 98.593162 = \$22{,}788.32$$

The slight difference is due to rounding.

Solution 3.5

Since the difference between the payments is always \$20, and the last payment is \$600 less 10 decrements of \$20, the last payment must be at time 11 years since the first payment was at time 1 year.

	600	580	560	...	400	payment
0	1	2	3	...	11	time
	220	200	180	...	20	decreasing payment
	380	380	380	...	380	level payment

The present value is:

$$600v + 580v^2 + 560v^3 + \cdots + 400v^{11} = 20(Da)_{\overline{11}|4\%} + 380a_{\overline{11}|4\%}$$

Calculating the required values:

$$a_{\overline{11}|4\%} = \frac{1 - 1.04^{-11}}{0.04} = 8.760476711$$

$$\Rightarrow \quad (Da)_{\overline{11}|4\%} = \frac{11 - 8.760476711}{0.04} = 55.98808223$$

So the present value is:

$$20 \times 55.98808223 + 380 \times 8.760476711 = \$4{,}448.74$$

Alternatively, the present value could also have been expressed as $620a_{\overline{11}|4\%} - 20(Ia)_{\overline{11}|4\%}$. However it is somewhat easier to calculate $(Da)_{\overline{11}|4\%}$ and $a_{\overline{11}|4\%}$ together, rather than $(Ia)_{\overline{11}|4\%}$ and $a_{\overline{11}|4\%}$.

Solution 3.6

Since we already calculated the present value at time 0 in Solution 3.5, we can accumulate the present value for 20 years. The accumulated value at 20 years is:

$$4,448.74(1.04)^{20} = \$9,747.74$$

Alternatively, working from first principles, the accumulated value at time 11 years is:

$$600(1+i)^{10} + 580(1+i)^{9} + 560(1+i)^{8} + \cdots + 400 = 20(Ds)_{\overline{11}|4\%} + 380s_{\overline{11}|4\%}$$

Calculating the required values:

$$s_{\overline{11}|4\%} = \frac{1.04^{11}-1}{0.04} = 13.486351$$

$$\Rightarrow \quad (Ds)_{\overline{11}|4\%} = \frac{11(1.04)^{11} - 13.486351}{0.04} = 86.191080$$

So the accumulated value at 11 years is:

$$20 \times 86.191080 + 380 \times 13.486351 = 6,848.635$$

The accumulated value at 20 years is:

$$6,848.635(1.04)^{9} = \$9,747.74$$

Solution 3.7

The timeline diagram of the payments is:

50	40	30	20	10	20	30	40	50	payment
0	1	2	3	4	5	6	7	8	time

The present value of the payments at time 0 is:

$$50 + 40v + 30v^2 + 20v^3 + \cdots + 40v^7 + 50v^8$$

The payments from time 0 to 4 years make up a decreasing annuity-due, so let's treat them separately from the rest. The present value of the first 5 payments is:

$$50 + 40v + 30v^2 + 20v^3 + 10v^4 = 10(D\ddot{a})_{\overline{5}|5\%}$$

Calculating the required values for the first 5 payments:

$$a_{\overline{5}|5\%} = \frac{1-1.05^{-5}}{0.05} = 4.329477$$

$$\Rightarrow \quad (D\ddot{a})_{\overline{5}|5\%} = \frac{5 - 4.329477}{0.05/1.05} = 14.080990$$

So the present value of the first 5 payments is:

$$10 \times 14.080990 = 140.80990$$

The payments from time 5 to 8 years almost match those of an increasing annuity-due with payments starting at time 4 years, but we used the \$10 payment at time 4 to value the first 5 payments. The increasing annuity-due is missing \$10 at time 4 years, so we can subtract it from the present value of an increasing annuity-due to get the present value of the last 4 payments from time 5 to 8 years. The present value at time 4 years of the last 4 payments is:

$$20v + 30v^2 + 40v^3 + 50v^4 = 10(I\ddot{a})_{\overline{5}|5\%} - 10$$

Calculating the required values for the last 4 payments:

$$\ddot{a}_{\overline{5}|5\%} = \frac{1-1.05^{-5}}{0.05/1.05} = 4.545951$$

$$\Rightarrow \quad (I\ddot{a})_{\overline{5}|5\%} = \frac{4.545951 - 5(1.05)^{-5}}{0.05/1.05} = 13.194713$$

So the present value of the last 4 payments at time 4 years is:

$$10 \times 13.194713 - 10 = 121.9471$$

The present value of the last 4 payments at time 0 is:

$$121.9471(1.05)^{-4} = 100.3262$$

Summing the present value of the first 5 payments with that of the last 4 payments, we have:

$$140.8090 + 100.3262 = \$241.14$$

Solution 3.8

The accumulated value at 10 years needs to equal \$750,000:

$$750,000 = X(1+i)^{10} + (X-5,000)(1+i)^9 + (X-10,000)(1+i)^8 + \cdots + (X-45,000)(1+i)$$
$$750,000 = (X-50,000)\ddot{s}_{\overline{10}|5\%} + 5,000(D\ddot{s})_{\overline{10}|5\%}$$

Calculating the required values:

$$s_{\overline{10}|5\%} = \frac{1.05^{10}-1}{0.05} = 12.577893$$
$$\ddot{s}_{\overline{10}|5\%} = 12.577893(1.05) = 13.206787$$
$$\Rightarrow \quad (D\ddot{s})_{\overline{10}|5\%} = \frac{10(1.05)^{10} - 12.577893}{1-1.05^{-1}} = 77.932119$$

Working with the equation of value, we have:

$$750,000 = (X-50,000)(13.206787) + 5,000(77.932119)$$
$$X = \$77,284.41$$

Alternatively, the present value of the investments could be determined first and then accumulated for 10 years to determine the accumulated value. The present value is:

$$X + (X-5,000)v + (X-10,000)v^2 + \cdots + (X-45,000)v^9$$
$$= (X-50,000)\ddot{a}_{\overline{10}|5\%} + 5,000(D\ddot{a})_{\overline{10}|5\%}$$

So the accumulated value of the investments is:

$$(1+i)^{10}\left\{(X-50,000)\ddot{a}_{\overline{10}|5\%} + 5,000(D\ddot{a})_{\overline{10}|5\%}\right\}$$

This needs to be \$750,000, which implies:

$$(1+i)^{10}\left\{(X-50,000)\ddot{a}_{\overline{10}|5\%}+5,000(D\ddot{a})_{\overline{10}|5\%}\right\}=750,000$$

Calculating the required values:

$$\ddot{a}_{\overline{10}|5\%}=\frac{1-1.05^{-10}}{1-1.05^{-1}}=8.10782168$$

$$a_{\overline{10}|5\%}=\frac{1-1.05^{-10}}{0.05}=7.721734929$$

$$\Rightarrow \quad (D\ddot{a})_{\overline{10}|5\%}=\frac{10-7.721734929}{1-1.05^{-1}}=47.8435665$$

So:

$$1.05^{10}\left\{X\ddot{a}_{\overline{10}|5\%}-405,391.0838+239,217.8325\right\}=750,000$$

$$\Rightarrow \quad X\ddot{a}_{\overline{10}|5\%}=\frac{750,000}{1.05^{10}}+405,391.0838-239,217.8325$$

$$\Rightarrow \quad X\ddot{a}_{\overline{10}|5\%}=626,608.1915$$

$$\Rightarrow \quad X=\$77,284.41$$

Alternatively, we could use:

$$X\ddot{s}_{\overline{10}|5\%}-5,000(Is)_{\overline{9}|5\%}(1+i)=750,000 \qquad \text{or} \qquad (1+i)^{10}\left\{X\ddot{a}_{\overline{10}|5\%}-5,000(Ia)_{\overline{9}|5\%}\right\}=750,000$$

or even:

$$(X+5,000)\ddot{s}_{\overline{10}|5\%}-5,000(I\ddot{s})_{\overline{10}|5\%}=750,000 \qquad \text{or} \qquad (1+i)^{10}\left\{(X+5,000)\ddot{a}_{\overline{10}|5\%}-5,000(I\ddot{a})_{\overline{10}|5\%}\right\}=750,000$$

Solution 3.9

The present value of the payments at time 6 years is:

$$15,000v+14,000v^2+\cdots+Yv^8$$

where Y is the last payment, which we can calculate:

$$Y=15,000-1,000\times 7=8,000$$

		15	14	13	...	8	payment (000's)
0	6	7	8	9	...	14	time
		8	7	6	...	1	decreasing payment (000's)
		7	7	7	...	7	level payment (000's)

So the present value at time 6 years is:

$$15,000v+14,000v^2+\cdots+8,000v^8=1,000(Da)_{\overline{8}|8\%}+7,000a_{\overline{8}|8\%}$$

The present value at time 0 is:

$$v^6\left(1,000(Da)_{\overline{8}|8\%}+7,000a_{\overline{8}|\%}\right)$$

So, $X = v^6\left(1{,}000(Da)_{\overline{8|}8\%} + 7{,}000a_{\overline{8|}8\%}\right)$.

Calculating the required values:

$$a_{\overline{8|}8\%} = \frac{1-1.08^{-8}}{0.08} = 5.746638944$$

$$\Rightarrow \quad (Da)_{\overline{8|}8\%} = \frac{8-5.746638944}{0.08} = 28.1670132$$

So:

$$X = 1.08^{-6}(1{,}000\times 28.1670132 + 7{,}000\times 5.746638944) = \$43{,}099.50$$

Alternatively, the value at time 6 years could be obtained using:

$$16{,}000a_{\overline{8|}8\%} - 1{,}000(Ia)_{\overline{8|}8\%}$$

Solution 3.10

The present value of the payments at time 7 years is:

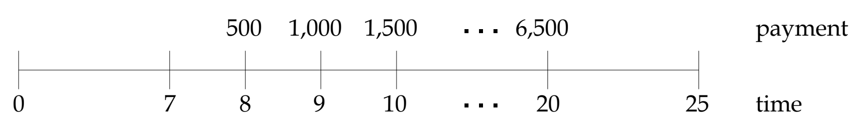
$$500v + 1{,}000v^2 + 1{,}500v^3 \cdots + Yv^{13}$$

where Y is the last payment, which we can calculate:

$$Y = 500 + 500\times 12 = 6{,}500$$

So the present value at time 7 years is:

$$500v + 1{,}000v^2 + 1{,}500v^3 \cdots + 6{,}500v^{13} = 500(Ia)_{\overline{13|}4.5\%}$$



The accumulated value at time 25 years is:

$$(1+i)^{18}\,500(Ia)_{\overline{13|}4.5\%}$$

Calculating the required values:

$$\ddot{a}_{\overline{13|}4.5\%} = \frac{1-1.045^{-13}}{1-1.045^{-1}} = 10.11858078$$

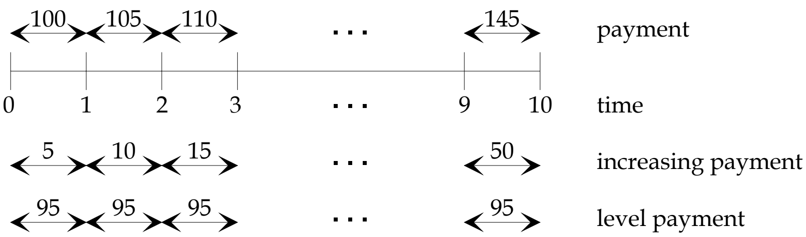
$$\Rightarrow \quad (Ia)_{\overline{13|}4.5\%} = \frac{10.11858078 - 13\times 1.045^{-13}}{0.045} = 61.8455433$$

So the accumulated value is:

$$500(1.045)^{18}(61.8455433) = \$68{,}292.28$$

Solution 3.11

These payments are equivalent to a level, continuously payable level annuity with payments of \$95 each year, plus \$5 times a continuously payable increasing annuity.



The accumulated value at 10 years is:

$$95\bar{s}_{\overline{10}|7\%} + 5(I\bar{s})_{\overline{10}|7\%}$$

Calculating the required values, we have:

$$\ddot{s}_{\overline{10}|7\%} = \frac{1.07^{10} - 1}{1 - 1.07^{-1}} = 14.783599$$

$$\bar{s}_{\overline{10}|7\%} = \frac{1.07^{10} - 1}{\ln(1.07)} = 14.294571$$

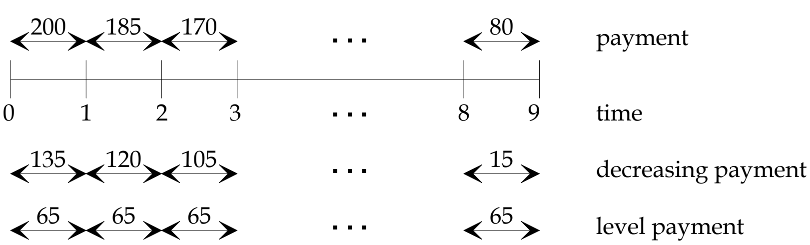
$$(I\bar{s})_{\overline{10}|7\%} = \frac{14.783599 - 10}{\ln(1.07)} = 70.701964$$

So the accumulated value at 10 years is:

$$95 \times 14.294571 + 5 \times 70.701964 = \$1{,}711.49$$

Solution 3.12

These payments are equivalent to a level, continuously payable 9-year annuity with payments of \$65 each year, plus \$15 times a continuously payable 9-year decreasing annuity.



The accumulated value at 9 years is:

$$65\bar{s}_{\overline{9}|4\%} + 15(D\bar{s})_{\overline{9}|4\%}$$

Calculating the required values, we have:

$$s_{\overline{9}|4\%} = \frac{1.04^9 - 1}{0.04} = 10.582795$$

$$\bar{s}_{\overline{9}|4\%} = \frac{1.04^9 - 1}{\ln(1.04)} = 10.793068$$

$$(D\bar{s})_{\overline{9}|4\%} = \frac{9(1.04)^9 - 10.582795}{\ln(1.04)} = 56.781502$$

So the accumulated value at 9 years is:

$$65 \times 10.793068 + 15 \times 56.781502 = 1,553.2719$$

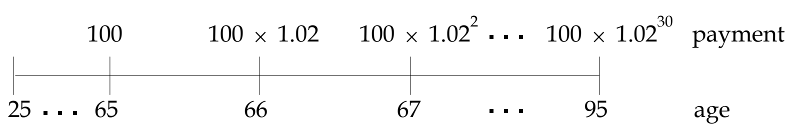
The accumulated value at 10 years is:

$$1,553.2719(1.04) = \$1,615.40$$

Solution 3.13

Let's first use first principles to answer this question, then we'll use an annuity-immediate as instructed. There are $96 - 65 + 1 = 31$ payments.

The timeline diagram of the payments (expressed in \$1,000's) is:



Since we're using an annuity-immediate, the present value at age 64 (in \$1,000's) is:

$$\frac{100}{1.06} + \frac{100(1.02)}{1.06^2} + \frac{100(1.02)^2}{1.06^3} + \cdots + \frac{100(1.02)^{30}}{1.06^{31}}$$

This expression can be simplified and determined using a geometric series:

$$\frac{100}{1.06}\left[1 + \frac{1.02}{1.06} + \left(\frac{1.02}{1.06}\right)^2 + \cdots + \left(\frac{1.02}{1.06}\right)^{30}\right] = \frac{100}{1.06}\left[\frac{1-\left(1.02/1.06\right)^{31}}{1 - 1.02/1.06}\right]$$

$$= \$1,741.31148$$

Using the formula for a compound increasing annuity-immediate, we first need to calculate j:

$$j = \frac{i-e}{1+e} = \frac{0.06 - 0.02}{1.02} = 3.921569\%$$

The present value of the payments at age 64 (in \$1,000's) is:

$$\frac{100}{1.02} a_{\overline{31}|j} = \frac{100}{1.02}\left[\frac{1-(1.03921569)^{-31}}{0.03921569}\right] = 1,741.31150$$

The slight difference is due to rounding. The present value at age 64 needs to be discounted by 39 years to get the present value at age 25. The present value at age 25 is:

$$1,000 \times 1,741.31150 \times 1.06^{-39} = \$179,451.76$$

Solution 3.14

Using the formula for a compound increasing annuity-due, we use the same interest rate j as before:

$$j = \frac{i-e}{1+e} = \frac{0.06-0.02}{1.02} = 3.921569\%$$

Since we're using an annuity-due, the present value of the payments at age 65 (in \$1,000's) is:

$$100\ddot{a}_{\overline{31}|j} = 100\left[\frac{1-(1.039216)^{-31}}{0.039216/1.039216}\right] = 1,845.790174$$

The present value at age 65 needs to be discounted by 40 years to get the present value at age 25. The present value at age 25 is:

$$1,000 \times 1,845.790174 \times 1.06^{-40} = \$179,451.76$$

Solution 3.15

The present value is:

$$\begin{aligned}
&\int_0^{10} (5t+1)\exp\left[-\int_0^t 0.01+0.05s \; ds\right] dt \\
&= \int_0^{10} (5t+1)\exp\left\{-\left[0.01s+0.025s^2\right]_0^t\right\} dt \\
&= \int_0^{10} (5t+1)\exp\left[-0.01t-0.025t^2\right] dt
\end{aligned}$$

Since $\frac{d}{dt}(-0.01t-0.025t^2) = -0.01-0.05t = \frac{5t+1}{-100}$, the above integral can be written in the following form (which can then be evaluated directly):

$$\begin{aligned}
&-100\int_0^{10} (-0.01-0.05t)\exp\left[-0.01t-0.025t^2\right] dt \\
&= -100\left[\exp\left\{-0.01t-0.025t^2\right\}\right]_0^{10} \\
&= -100(e^{-0.1-2.5}-e^0) \\
&= -100(e^{-2.6}-1) \\
&= \$92.57
\end{aligned}$$

Solution 3.16

The present value at time 5 years is:

$$\int_5^{10} (1.8t^2 + 6t)\exp\left[-\int_5^t (0.0003s^2 + 0.001s)\ ds\right] dt$$
$$= \int_5^{10} (1.8t^2 + 6t)\exp\left\{-\left[0.0001s^3 + 0.0005s^2\right]_5^t\right\} dt$$
$$= \int_5^{10} (1.8t^2 + 6t)\exp\left[-0.0001t^3 - 0.0005t^2 + 0.025\right] dt$$

Since $\frac{d}{dt}(-0.0001t^3 - 0.0005t^2 + 0.025) = -0.0003t^2 - 0.001t = \frac{1.8t^2 + 6t}{-6,000}$, the above integral can be written in the following form (which can then be evaluated directly):

$$-6,000\int_5^{10} (-0.0003t^2 - 0.001t)\exp\left[-0.0001t^3 - 0.0005t^2 + 0.025\right] dt$$
$$= -6,000\left[\exp\left[-0.0001t^3 - 0.0005t^2 + 0.025\right]\right]_5^{10}$$
$$= -6,000(e^{-0.1-0.05+0.025} - e^{-0.0125-0.0125+0.025})$$
$$= 705.01858$$

The present value at time 0 is:

$$705.01858\exp\left[-\int_0^5 0.008t + 0.03\ dt\right]$$
$$= 705.01858\exp\left\{-\left[0.004t^2 + 0.03t\right]_0^5\right\}$$
$$= 705.01858\exp\{-0.1 - 0.15\}$$
$$= \$549.07$$

Solution 3.17

When interest is credited into another account that is reinvested at another interest rate, the payment pattern fits the pattern of an increasing annuity.

Susan makes her first investment of Z at the end of year 1. The 5% interest on that investment is paid at the end of year 2, at which time it is invested for 5 years at 6%. This is represented in the first term of the equation below.

She makes her second investment of Z at the end of year 2. The 5% interest on that investment, and on the original investment, is paid at the end of year 3, at which time it is invested for 4 years at 6%. This is represented in the second term below.

The process continues until we get to her last payment. Her last payment does not earn any interest. Adding it to the other 6 payments gives us 7Z, as shown in the last term:

$$X = Z(1)(0.05)(1.06)^5 + Z(2)(0.05)(1.06)^4 + \cdots + Z(5)(0.05)(1.06)^1 + Z(6)(0.05)(1.06)^0 + 7Z$$

We can simplify the equation:

$$X = Z(0.05)[1(1.06)^5 + 2(1.06)^4 + \cdots + 5(1.06)^1 + 6(1.06)^0] + 7Z$$

$$X = Z(0.05)(Is)_{\overline{6|}0.06} + 7Z$$

We know that $(Is)_{\overline{n|}} = \dfrac{\ddot{s}_{\overline{n|}} - n}{i}$.

$$X = Z(0.05)\frac{\ddot{s}_{\overline{6|}0.06} - 6}{0.06} + 7Z$$

We also know that $\ddot{s}_{\overline{n|}} = \dfrac{(1+i)^n - 1}{d}$, and $d = \dfrac{i}{1+i} = \dfrac{0.06}{1.06} = 0.05660$.

So, we have:

$$X = Z(0.05)\frac{\dfrac{(1.06)^6 - 1}{0.05660} - 6}{0.06} + 7Z$$

$$X = Z(0.05)\frac{7.3938 - 6}{0.06} + 7Z$$

$$X = 8.1615Z$$

Using the same logic, we can find Y:

$$Y = Z(0.025)[1(1.03)^{12} + 2(1.06)^{11} + \cdots + 12(1.03)^1 + 13(1.03)^0] + 14Z$$

$$Y = Z(0.025)(Is)_{\overline{13|}0.03} + 14Z$$

$$Y = Z(0.025)\frac{\dfrac{(1.03)^{13} - 1}{0.029126} - 13}{0.03} + 14Z$$

$$Y = Z(0.025)\frac{16.08632 - 13}{0.03} + 14Z$$

$$Y = 16.5719Z$$

It is now a simple matter to find the ratio of Y to X:

$$\frac{Y}{X}=\frac{16.5719Z}{8.1615Z}=2.0305$$

Solution 3.18

The perpetuity can be split into two parts: a 5-year annuity-immediate and a perpetuity-immediate that begins in 5 years. Since we are given the value of the perpetuity, we can solve for the growth rate.

The perpetuity-immediate pays 10 at the end of every year for 5 years, after which the payments increase by $k\%$ per year. We can split this perpetuity into two parts: a 5-year annuity-immediate of 10 and a perpetuity-immediate that pays $10(1+k)$ at the end of 6 years with payments that increase by $k\%$ each year. We are given the interest rate is 9.2%. The preset value of the 5-year annuity-immediate equals:

$$10a_{\overline{5}|9.2\%}=10\frac{1-v_{9.2\%}^{5}}{0.092}=10\frac{1-\left(\frac{1}{1.092}\right)^{5}}{0.092}=10(3.86955)=38.6955$$

The perpetuity-immediate begins in 5 years, so it must be discounted by 5 years at 9.2%. The present value of the perpetuity-immediate is:

$$\begin{aligned}&v^{5}(10)\left[\frac{1+k}{1.092}+\left(\frac{1+k}{1.092}\right)^{2}+\left(\frac{1+k}{1.092}\right)^{3}+\cdots\right]\\&=10v^{5}\frac{1+k}{1.092}\left[1+\left(\frac{1+k}{1.092}\right)^{1}+\left(\frac{1+k}{1.092}\right)^{2}+\cdots\right]\end{aligned}$$

The part in the brackets is a geometric series with a ratio term of $\frac{1+k}{1.092}$ and n equal to infinity. This geometric series simplifies to:

$$\frac{1-\left(\frac{1+k}{1.092}\right)^{\infty}}{1-\frac{1+k}{1.092}}=\frac{1}{1-\frac{1+k}{1.092}}=\frac{1}{\frac{1.092-1-k}{1.092}}=\frac{1.092}{0.092-k}$$

We are given that the present value of the initial perpetuity equals 167.50. Putting all the pieces together, we have:

$$\begin{aligned}167.50&=38.6955+\frac{10}{(1.092)^{5}}\times\frac{1+k}{1.092}\times\frac{1.092}{0.092-k}\\128.8045&=6.44001\times\frac{(1+k)}{1.092}\times\frac{1.092}{0.092-k}\\20.00&=\frac{1+k}{0.092-k}\\1.84-20k&=1+k\\21k&=0.84\\k&=0.04\end{aligned}$$

Solution 3.19

This is a continuously varying payment stream with a varying force of interest. We are given the accumulated value of this stream at time 10 years. We set up the equation of value and solve for k, a factor within the payment stream function.

The general formula for the accumulated value at time b from time a of a varying payment stream function ρ_t at a force of interest function δ_s is:

$$\int_a^b \rho_t \exp\left[\int_t^b \delta_s \, ds\right] dt$$

In this case, the continuously varying payment stream is the function $\rho_t = (8k + tk)$. The payments are received from time $t = a = 0$ to time $t = b = 10$. The varying force of interest is the function $\delta_t = \dfrac{1}{8+t}$. We are given that the accumulated value equals \$20,000. We set up the equation of value:

$$20{,}000 = \int_0^{10} (8k + tk)\exp\left(\int_t^{10} \frac{1}{8+s} ds\right) dt$$

Let's solve this equation in parts. The first part is:

$$\exp\left(\int_t^{10} \frac{1}{8+s} ds\right) = \exp\left(\ln(8+s)\Big|_t^{10}\right) = \exp\left[\ln(8+10) - \ln(8+t)\right] = \exp\left[\ln\left(\frac{18}{8+t}\right)\right] = \frac{18}{8+t}$$

The first part can be inserted into the equation of value to get:

$$20{,}000 = \int_0^{10} (8k + tk)\frac{18}{8+t} dt = \int_0^{10} 18k\frac{8+t}{8+t} dt = \int_0^{10} 18k dt = 18kt\Big|_0^{10} = 180k$$

$$k = \frac{20{,}000}{180} = 111.11$$

Solution 3.20

Solution 3.21

Let's separate the payment stream into two parts. The first part is a level annuity-immediate of \$60 at the end of each year for 20 years. The second part begins with a payment of \$140 at time 1 year, after which the payments decrease by \$20 per year. Since the difference between the payments in the second part is always \$20, the last payment is \$20 at time 7 years.

	200	180	160	. . .	60	60	. . .	60	payment
0	1	2	3	. . .	7	8	. . .	20	time
	140	120	100	. . .	20	0	. . .	0	decreasing payment
	60	60	60	. . .	60	60	. . .	60	level payment

The present value is:

$$200v + 180v^2 + 160v^3 + \cdots + 60v^{20} = 20(Da)_{\overline{7}|4\%} + 60a_{\overline{20}|4\%}$$

Calculating the required values:

$$a_{\overline{20}|4\%} = \frac{1-1.04^{-20}}{0.04} = 13.590326$$

$$a_{\overline{7}|4\%} = \frac{1-1.04^{-7}}{0.04} = 6.002055$$

$$\Rightarrow \quad (Da)_{\overline{7}|4\%} = \frac{7-6.002055}{0.04} = 24.948633$$

So the present value is:

$$20 \times 24.948633 + 60 \times 13.590326 = \$1,314.39$$

Alternatively, the present value could also have been expressed as $200a_{\overline{20}|4\%} - 20v(Ia)_{\overline{7}|4\%} - 140v^8 a_{\overline{12}|4\%}$. However it is somewhat easier to calculate $(Da)_{\overline{7}|4\%}$ and $a_{\overline{20}|4\%}$.