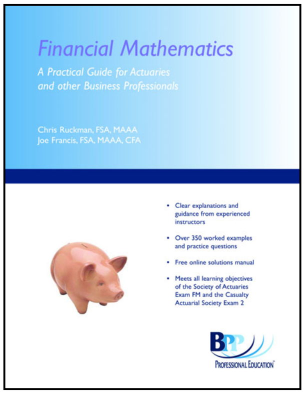

Financial Mathematics

A Practical Guide for Actuaries and other Business Professionals

By Chris Ruckman, FSA & Joe Francis, FSA, CFA

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Solutions to practice questions – Chapter 6

Solution 6.1

The annual effective yield on the 13-week T-bill is 6.5%. Setting up the equation of value and solving for the unknown, we have:

$$\left(\frac{1,000}{X}\right)^{\frac{52}{13}} - 1 = 6.5\%$$

$$\frac{1,000}{X} = 1.015868$$

$$X = \$984.38$$

Solution 6.2

The investor deposits $1,000,000 at the start of each year in a 5-year GIC with an insurance company that guarantees an annual 6% interest rate on the fund. If the interest that is generated each year is reinvested at 6%, the lump sum payable to the investor at 5 years is:

$$1,000,000\ddot{s}_{\overline{5}|6\%} = 1,000,000\frac{(1.06)^5 - 1}{0.06/1.06} = \$5,975,318.54$$

However, if the 6% guaranteed interest generated from the GIC is reinvested at a 4% annual rate instead of a 6% annual rate, the amount payable to the investor at 5 years is:

$$5 \times 1,000,000 + 60,000(1.04)^4 + 2 \times 60,000(1.04)^3 + 3 \times 60,000(1.04)^2 + 4 \times 60,000(1.04) + 5 \times 60,000$$

$$\Rightarrow 5,000,000 + 60,000(Is)_{\overline{5}|4\%}$$

Calculating the required values, we have:

$$\ddot{s}_{\overline{5}|4\%} = \frac{(1.04)^5 - 1}{0.04/1.04} = 5.632975$$

$$(Is)_{\overline{5}|4\%} = \frac{5.632975 - 5}{0.04} = 15.824387$$

The amount payable to the investor after 5 years is:

$$5{,}000{,}000 + 60{,}000(15.824387) = \$5{,}949{,}463.19$$

Solution 6.3

The table below shows the cash flows received and paid by both Company A and Company B. Only the net cash flow payments are actually made.

When LIBOR is above 6.8%, the fixed-rate payer receives payments from the dealer, shown as positive numbers in the fifth column below. When LIBOR is below 6.8%, the fixed-rate payer makes payments to the dealer, shown as negative numbers in the fifth column.

		Company A			**Company B**			
Time	1-year LIBOR	Receives	Pays	**Net Payment Received**	Receives	Pays	**Net Payment Received**	Dealer Receives
1	7.0%	70,000	68,000	**2,000**	67,000	70,000	**-3,000**	1,000
2	8.0%	80,000	68,000	**12,000**	67,000	80,000	**-13,000**	1,000
3	6.7%	67,000	68,000	**-1,000**	67,000	67,000	**0**	1,000
4	6.0%	60,000	68,000	**-8,000**	67,000	60,000	**7,000**	1,000

When LIBOR is above 6.7%, the floating-rate payer makes payments to the dealer, shown as negative numbers in the eighth column. When LIBOR is below 6.7%, the floating-rate payer receives payments from the dealer, shown as positive numbers in the eighth column.

Regardless of LIBOR's level, the dealer receives payments of \$1,000, which is equal to 10 basis points times the notional amount of \$1,000,000.

Solution 6.4

There are no cash flows at 6 months nor at one year since the index rate is greater than or equal to the strike rate at these times.

The first cash flow from the floor occurs at 18 months:

$$\text{Floor payment} = \text{Max}\left[\frac{4.0\% - 3.5\%}{2}, 0\right] \times 1{,}000{,}000 = \$2{,}500.00$$

The second cash flow from the floor occurs at 2 years:

$$\text{Floor payment} = \text{Max}\left[\frac{4.0\% - 2.75\%}{2}, 0\right] \times 1{,}000{,}000 = \$6{,}250.00$$

Solution 6.5

We have:

$$F = 100; \quad C = 105; \quad r = \frac{0.04}{2} = 2.0\%; \quad g = 0.02 \times \frac{100}{105} = 0.19048\%$$

$$i = \frac{0.05}{2} = 2.5\%$$

$$n = 2 \times 15 = 30 \text{ coupons}; \quad Fr = 100(0.02) = 2$$

Working in semiannual periods, the price is:

$$P = 2a_{\overline{30}|2.5\%} + 105(1.025)^{-30} = 2(20.930293) + 105(0.476743) = \$91.92$$

Solution 6.6

Immediately after the 10th coupon is paid, there are 20 coupons left. Working in semiannual periods, the book value is:

$$BV_{10} = 2a_{\overline{20}|2.5\%} + 105(1.025)^{-20} = 2(15.589162) + 105(0.610271) = \$95.26$$

Solution 6.7

Using the book value immediately after the 10th coupon is paid, the interest and principal portions of the 11th coupon can be determined:

$$I_{11} = iBV_{10} = 0.025(95.26) = \$2.38$$
$$PA_{11} = I_{11} - Fr = 2.38 - 2.00 = \$0.38$$

Alternatively, the amount of principal amortized (since this is a discount bond, this is the amount of discount accumulation) in the 11th coupon can be determined directly:

$$g = 2/105 = 0.019048$$
$$PA_{11} = C(g-i)v^{n-t+1} = 105(0.019048 - 0.025)(1.025)^{-(30-11+1)} = -\$0.38$$
$$\text{Discount accumulation}_{11} = -PA_{11} = -(-0.38) = 0.38$$

Solution 6.8

Given the principal adjustment amounts in two different coupons, the periodic effective yield is determined by dividing the latter amount by the former amount:

$$\frac{PA_{15}}{PA_{10}} = \frac{0.345233}{0.305135} = (1+i)^5$$
$$i = (1.131411)^{\frac{1}{5}} - 1 = 2.5\%$$

This interest rate is the semiannual effective yield since the coupons are paid semiannually. The annual effective yield is:

$$i = (1.025)^2 - 1 = 5.06\%$$

Solution 6.9

We have:

$$n = 2\text{x}10 = 20 \text{ coupons}; \quad F = 1{,}000; \quad r = \frac{0.10}{2} = 5\%; \quad Fr = 50$$
$$i = 0.03$$

We first need to find the redemption payment. Working in half-years, the per \$1,000 face amount is:

$$P = 1{,}235 = 50a_{\overline{20}|} + Cv^{20}$$

This equation is evaluated at the semiannual effective yield of 3%.

So we have:

$$1{,}235 = 50\frac{(1-1.03^{-20})}{0.03} + C(1.03)^{-20} \quad \Rightarrow \quad C = \$887.03$$

With 12 coupons left, the book value immediately after the 8th payment is:

$$BV_8 = 50a_{\overline{12|}} + 887.03v^{12} = \frac{50(1-1.03^{-12})}{0.03} + 887.03(1.03)^{-12} = \$1,119.84$$

So the interest earned in the 9th payment is:

$$I_9 = iBV_8 = 0.03 \times 1,119.84 = \$33.60$$

Solution 6.10

Let c be the semiannual coupon amount. Solving the bond price equation of value for the coupon amount, we have:

$$884.426140 = ca_{\overline{40|}3\%} + 1,000(1.03)^{-40} = c\frac{1-1.03^{-40}}{0.03} + 1,000(0.306557)$$

$$c = \frac{884.426140 - 306.556841}{23.114772} = \$25.00$$

Solution 6.11

Solving the level dividend stock price formula for the required return, we have:

$$PV \text{ stock} = \frac{div_1}{r}$$

$$r = \frac{div_1}{P_0} = \frac{10}{200} = 0.05$$

Solution 6.12

Solving the constant dividend growth price formula for the dividend growth rate, we have:

$$PV \text{ stock} = \frac{div_1}{r-g}$$

$$g = r - \frac{div_1}{P_0} = 0.05 - \frac{10}{333.33} = 0.02$$

Solution 6.13

This stock has a level quarterly coupon of \$0.50 and is sold after 5 years, or 20 quarters. This bond can be valued like a bond that pays quarterly dividends, where the stock's sale price is analogous to the bond's redemption amount. The quarterly effective yield is:

$$\frac{i^{(4)}}{4} = (1.15)^{\frac{1}{4}} - 1 = 0.035558$$

Working in quarters, we have:

$$P = 0.50a_{\overline{20|}3.5558\%} + 15(1.035558)^{-20} = \frac{0.50(1-1.035558^{-20})}{0.035558} + 15(1.035558)^{-20} = \$14.53$$

Solution 6.14

The stock price is determined by multiplying the earnings per share times the price-to-earnings ratio. The EPS is the same as net income per share.

$$PV \text{ stock} = P_0 = (\text{P/E ratio})(\text{net income per share})$$
$$= 12.1\left(\frac{25}{1.5}\right) = \$201.67$$

Solution 6.15

Jack sells 100 shares short for \$60 per share and he buys them back for \$50 per share, so his capital gain is $1{,}000(60-50)=\$1{,}000$. Jack's margin deposit is $0.50\times 1{,}000(60)=\$3{,}000$, and he earns 5% on the margin deposit. The stock paid a dividend of $\$X$ per share. Rearranging the short sale yield equation, we can solve for X:

$$0.316667 = \frac{1{,}000+(3{,}000)(0.05)-100X}{3{,}000}$$
$$X = \frac{1{,}000+150-3{,}000(0.316667)}{100} = \$2$$

Jill sells 100 shares short for \$65 per share and buys them back for $\$B$ per share. Jill's margin deposit is $0.50\times 100(65)=\$3{,}250$, and she also earns 5% on the margin deposit. Using this information, we can determine the price at which Jill buys back the stock:

$$\frac{0.316667}{2} = \frac{(6{,}500-100B)+(3{,}250)(0.05)-200}{3{,}250}$$
$$100B = 6{,}500+162.50-200-0.158334(3{,}250)$$
$$B = \$59.48$$

Solution 6.16

Put-call parity can be used to find the value of S_0 in terms of $PV(X)$:

$$C_0 + PV(X) = P_0 + S_0$$
$$3.33 + PV(X) = 10.31 + S_0$$
$$S_0 = PV(X) - 6.98$$

The Black-Scholes formula is then used to determine the exercise price:

$$C_0 = S_0 N(d_1) - PV(X)N(d_2)$$
$$3.33 = S_0(0.366172) - PV(X)(0.289741)$$
$$3.33 = [PV(X)-6.98](0.366172) - PV(X)(0.289741) \qquad \text{(by substitution for } S_0\text{)}$$
$$PV(X) = 77.01$$
$$X = 77.01(1.08)^{0.5} = \$80.03$$

Solution 6.17

The 60-day risk-free discount factor is given by the return on the 60-day T-bill:

$$(1+r_f)^{-(T-t)} = \frac{993,333}{1,000,000} \qquad \Rightarrow \qquad (1+r_f)^{(T-t)} = \frac{1,000,000}{993,333}$$

The asset to be delivered in 60 days is a 90-day T-bill. Right now, that asset is the 150-day T-bill, so we have the current price of the asset to be delivered. Making use of the formula, the futures price is $985,739:

$$F(t,T) = S_t(1+r_f)^{(T-t)}$$
$$= 979,167 \times \frac{1,000,000}{993,333} = \$985,739$$

Note that in this case we didn't even need to calculate the value of the risk-free interest rate.

Solution 6.18

We have:

$$S_0 = 100 \,; \quad \sigma = 0.40 \,; \quad t = 0.25 \,; \quad \ln\left(\frac{100}{PV(X)}\right) = -0.08$$

This last bit of information can be used to find the present value of the exercise price:

$$\ln\left(\frac{100}{PV(X)}\right) = -0.08$$

$$\frac{100}{PV(X)} = 0.9231$$

$$PV(X) = 108.33$$

The first step to using the Black-Scholes formula is to calculate d_1 and d_2:

$$d_1 = \frac{\ln \frac{S_0}{PV(X)}}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2} = \frac{-0.08}{0.4\sqrt{0.25}} + \frac{0.4\sqrt{0.25}}{2} = -0.3$$

$$d_2 = d_1 - \sigma\sqrt{t} = -0.3 - 0.2 = -0.5$$

From the standard normal distribution table, we have:

$$N(d_1) = N(-0.3) = 0.3821$$

$$N(d_2) = N(-0.5) = 0.3085$$

The call price is $4.79:

$$C_0 = S_0 N(d_1) - PV(X)N(d_2) = 100(0.3821) - 108.33(0.3085) = 38.21 - 33.42 = 4.79$$

Solution 6.19

We have two bonds with the same redemption value but with different maturity dates. Bond X is a normal semiannual coupon bond, but bond Y is an accumulation bond, so it pays no coupons.

Let i be the semiannual yield. For bond X, we have:

$$381.50 = \frac{C}{(1+i)^{2n}}$$

Since bond Y is a zero-coupon bond, its price is equal to the present value of the redemption amount:

$$647.80 = \frac{C}{(1+i)^{n}}$$

Since the redemption amounts of the two bonds are equal, we divide the present value of the redemption amount of Bond Y by the redemption amount of Bond X to get an expression for $(1+i)^n$:

$$\frac{647.80}{381.50} = \frac{C/(1+i)^n}{C/(1+i)^{2n}}$$

$$(1+i)^n = 1.698034$$

Now we can determine the redemption amount:

$$C = 647.80(1.698034) = \$1,099.99$$

The price of bond X is:

$$\begin{aligned} P &= r(1,000)a_{\overline{2n}|} + 381.50 \\ &= (1.03125i)(1,000)\frac{1-v^{2n}}{i} + 381.50 \\ &= (1.03125)(1,000)\left[1-(1.698034)^{-2}\right] + 381.50 \\ &= \$1,055.09 \end{aligned}$$

Solution 6.20

The first dividend of \$5 occurs in 3 months. Subsequently, the dividends are paid annually and increase by 4% a year. The present value of this stream of nominal cash flows is:

$$5v^{0.25} + 5\times 1.04v^{1.25} + 5\times 1.04^2 v^{2.25} + 5\times 1.04^3 v^{3.25} + \cdots$$

We need to determine the real rate of return, so the nominal cash flows are converted to real cash flows. The present value of the real cash flows is:

$$\frac{5v^{0.25}}{1.015^{0.25}} + \frac{5\times 1.04v^{1.25}}{1.015^{1.25}} + \frac{5\times 1.04^2 v^{2.25}}{1.015^{2.25}} + \frac{5\times 1.04^3 v^{3.25}}{1.015^{3.25}} + \cdots$$

The stock price is \$125. We set this equal to the above equation, and simplify it using the infinite geometric series summation formula:

$$\begin{aligned}
125 &= \frac{5v^{0.25}}{1.015^{0.25}} + \frac{5\times 1.04v^{1.25}}{1.015^{1.25}} + \frac{5\times 1.04^2 v^{2.25}}{1.015^{2.25}} + \frac{5\times 1.04^3 v^{3.25}}{1.015^{3.25}} + \cdots \\
&= \frac{5v^{0.25}}{1.015^{0.25}}\left[1 + \frac{1.04v}{1.015} + \frac{1.04^2 v^2}{1.015^2} + \frac{1.04^3 v^3}{1.015^3} + \cdots\right] \\
&= 4.981424v^{0.25}\left[\frac{1}{1-1.024631v}\right] \\
&= 4.981424\frac{(1+i)^{0.75}}{(1+i)}\left[\frac{1}{1-1.024631v}\right] \\
&= \frac{4.981424(1+i)^{0.75}}{1+i-1.024631} \\
&= \frac{4.981424(1+i)^{0.75}}{i-0.024631}
\end{aligned}$$

Let's try plugging 6% in for i. When the annual effective real interest rate is 6%, the right-hand side equals \$147.131. Since this is greater than the \$125 we are trying to match, we need to increase the interest rate. When we plug 7% in for i, the right-hand side equals \$115.512. The interest rate we seek is in between these two values, so we can linearly interpolate to determine the interest rate:

$$0.07 - \frac{125.00 - 115.512}{147.131 - 115.512}(0.07 - 0.06) = 0.067$$

So the annual effective real interest rate is 6.7%.

