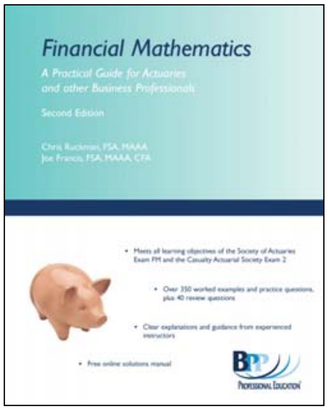

Financial Mathematics
Second Edition
A Practical Guide for Actuaries and other Business Professionals

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Solutions to practice questions – Chapter 2

Since Tyler's first deposit doesn't occur until his 23rd birthday, there are $65-23+1=43$ deposits. The accumulated value of these level deposits at time 43 years need to equal $1,000,000 on his 65th birthday:

$$Xs_{\overline{43|}4\%} = 1,000,000$$
$$X\left(\frac{(1+0.04)^{43}-1}{0.04}\right) = 1,000,000$$
$$X(110.012382) = 1,000,000$$
$$X = \$9,089.89$$

Solution 2.2

Paul's present value is:

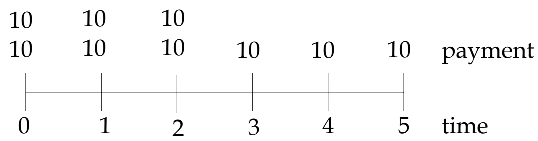
$$1,000a_{\overline{5|}5\%} = 1,000\left(\frac{1-(1.05)^{-5}}{0.05}\right)$$
$$= 1,000(4.329477)$$
$$= \$4,329.48$$

This is equal to Howard's present value, so we set up the equation of value and solve for X:

$$4,329.48 = Xa_{\overline{10|}5\%}$$
$$4,329.48 = X\left(\frac{1-(1.05)^{-10}}{0.05}\right)$$
$$4,329.48 = X(7.721735)$$
$$X = \$560.69$$

Solution 2.3

If we split the payments into two parts (a 3-year \$10 annuity-due and a six-year \$10 annuity-due), we get two payment streams that can be valued at time 0:



The 3-year annuity-due's present value is:

$$\begin{aligned}10\ddot{a}_{\overline{3}|8\%} &= 10\left(\frac{1-(1.08)^{-3}}{0.08/1.08}\right)\\ &= 10(2.783265)\\ &= \$27.83\end{aligned}$$

The 6-year annuity-due's present value is:

$$\begin{aligned}10\ddot{a}_{\overline{6}|8\%} &= 10\left(\frac{1-(1.08)^{-6}}{0.08/1.08}\right)\\ &= 10(4.992710)\\ &= \$49.93\end{aligned}$$

Since the present value of each part is valued at time 0, we can add their present values together. Adding the two pieces together, we have the present value of the entire cash flow stream:

$$27.83 + 49.93 = \$77.76$$

An alternate expression that produces the same result is:

$$20\ddot{a}_{\overline{3}|8\%} + 10\ddot{a}_{\overline{3}|8\%}v^3$$

Solution 2.4

The 25 deposits are made at the beginning of each year starting today. The first deposit is at time 0 and the last deposit occurs at time 24 years. The accumulated value of the 25-year annuity-due is valued at time 25 years:

$$\begin{aligned}10{,}000\ddot{s}_{\overline{25}|5.5\%} &= 10{,}000\left(\frac{(1.055)^{25}-1}{0.055/1.055}\right)\\ &= 10{,}000(53.965981)\\ &= \$539{,}659.81\end{aligned}$$

This accumulated value at 25 years is accumulated for 5 years to time 30 years. The time 30 year accumulated value is:

$$539{,}659.81(1.055)^5 = \$705{,}313.78$$

The accumulated value at 30 years is used to purchase a 25-year annuity-due. We solve for the unknown level payment of X:

$$X\ddot{a}_{\overline{25}|5.5\%} = 705,313.78$$

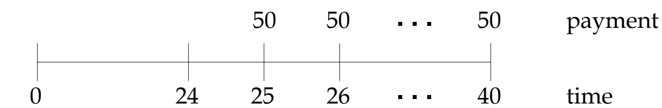
$$X\left(\frac{1-(1.055)^{-25}}{0.055/1.055}\right) = 705,313.78$$

$$X(14.151699) = 705,313.78$$

$$X = \$49,839.51$$

Solution 2.5

There are $40-25+1=16$ payments.



The present value at time 24 years is:

$$50a_{\overline{16}|12\%}$$

The present value at time 0 is:

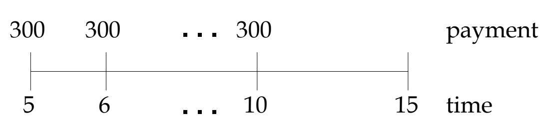
$$50v^{24}a_{\overline{16}|12\%} = 50(1.12)^{-24}\frac{(1-1.12^{-16})}{0.12} = \$22.97$$

Alternatively, we could use:

$$50v^{25}\ddot{a}_{\overline{16}|12\%} = 50(1.12)^{-25}\frac{(1-1.12^{-16})}{1-1.12^{-1}} = \$22.97$$

Solution 2.6

There are $10-5+1=6$ payments.



The accumulated value at time 10 years is:

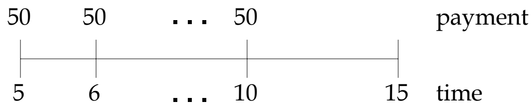
$$300s_{\overline{6}|3\%}$$

The accumulated value at time 15 years is:

$$300s_{\overline{6}|3\%}(1+i)^5 = 300\frac{(1.03^6-1)}{0.03}(1.03)^5 = \$2,249.60$$

Solution 2.7

There are $10-5+1=6$ payments.



The accumulated value at time 11 years is:

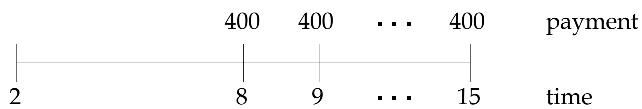
$$50\ddot{s}_{\overline{6}|16\%}$$

The accumulated value at time 15 years is:

$$50\ddot{s}_{\overline{6}|16\%}(1+i)^4 = 50\frac{1.16^6-1}{1-1.16^{-1}}(1.16)^4 = \$942.79$$

Solution 2.8

There are $15-8+1=8$ payments.



The present value at time 8 years is:

$$400\ddot{a}_{\overline{8}|9\%}$$

The present value at time 2 years is:

$$400\ddot{a}_{\overline{8}|9\%}v^6 = 400\frac{1-1.09^{-8}}{1-1.09^{-1}}(1.09)^{-6} = \$1,438.90$$

Alternatively, we could use:

$$400a_{\overline{8}|9\%}v^5 = 400\frac{1-1.09^{-8}}{0.09}(1.09)^{-5} = \$1,438.90$$

Solution 2.9

The period from time 5 years to time 6 years is the sixth year. The period of time from time 6 years to time 7 years is the seventh year. The period of time from time 7 years to time 8 years is the eighth year. We have three years of continuous payments from time 5 years to time 8 years. The continuous payment is $100 per year.

The accumulated value of this continuous annuity is valued at time 8 years:

$$\begin{aligned} 100\bar{s}_{\overline{3}|4.5\%} &= 100\left(\frac{(1.045)^3 - 1}{\ln(1.045)}\right) \\ &= 100(3.207090) \\ &= \$320.71 \end{aligned}$$

But the question asks for the accumulated value at time 10 years, so we need to accumulate the time 8 year accumulated value for another two years:

$$320.71(1.045)^2 = \$350.22$$

Solution 2.10

The present value of Mary's perpetuity-due is:

$$\begin{aligned} 100\ddot{a}_{\overline{\infty}|10\%} &= \frac{100}{0.10/1.10} \\ &= \$1,100.00 \end{aligned}$$

Virginia's perpetuity begins five years from today. If a perpetuity-immediate is used to value the payment series, the valuation date is at time 4 years. The present value of Virginia's perpetuity-immediate at time 4 years is $Xa_{\overline{\infty}|10\%}$, so we need to discount this amount for 4 years back to time 0 before it can be equated to Mary's present value.

After equating the discounted amount to Mary's present value, we can solve for X:

$$\begin{aligned} Xv^4 a_{\overline{\infty}|10\%} &= 1,100 \\ \frac{X(1.10)^{-4}}{0.10} &= 1,100 \\ X &= \$161.05 \end{aligned}$$

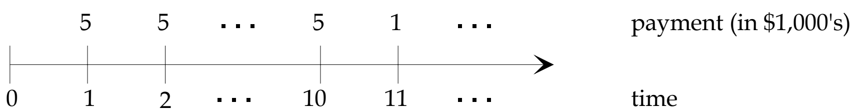
Alternatively, a perpetuity-due could be used to value Virginia's payment series. If a perpetuity-due is used to value the perpetuity, the valuation date is at time 5 years. The present value of Virginia's perpetuity-due at time 5 years is $X\ddot{a}_{\overline{\infty}|10\%}$, so we need to discount this amount for 5 years back to time 0 before it can be equated to Mary's present value.

After equating the discounted amount to Mary's present value, we can solve for X:

$$\begin{aligned} Xv^5 \ddot{a}_{\overline{\infty}|10\%} &= 1,100 \\ \frac{X(1.10)^{-5}}{0.10/1.10} &= 1,100 \\ X &= \$161.05 \end{aligned}$$

Solution 2.11

The timeline diagram of the payment stream is:



The present value of this payment stream can be split into two parts: a 10-year annuity-immediate of $5,000 followed by a perpetuity-immediate of $1,000 in which the first payment occurs at time 11.

The present value at time 0 of the 10-year annuity-immediate is:

$$5,000a_{\overline{10}|9\%} = 5,000\left(\frac{1-(1.09)^{-10}}{0.09}\right)$$
$$= 5,000(6.417658) = \$32,088.29$$

The present value at time 10 of the perpetuity-immediate is:

$$1,000a_{\overline{\infty}|9\%} = \frac{1,000}{0.09}$$
$$= \$11,111.11$$

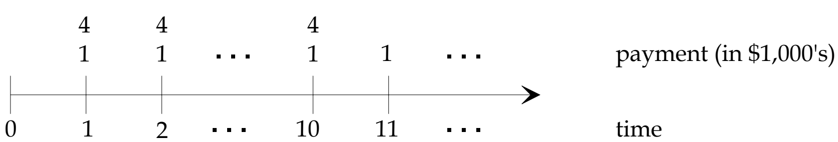
But this amount needs to be discounted back 10 years so that the present value is valued at time 0:

$$11,111.11v^{10} = 11,111.11(1.09)^{-10} = \$4,693.45$$

Summing the two time 0 present values, we have:

$$32,088.29 + 4,693.45 = \$36,781.74$$

Alternatively, the timeline diagram of the payment stream could be drawn this way:



The present value of this payment stream can be split into two parts: a 10-year annuity-immediate of $4,000 concurrent with a perpetuity-immediate of $1,000 in which the first payment occurs at time 1 year.

The present value at time 0 of the 10-year annuity-immediate is:

$$4,000a_{\overline{10}|9\%} = 4,000\left(\frac{1-(1.09)^{-10}}{0.09}\right)$$
$$= 4,000(6.417658) = \$25,670.63$$

This method results in the perpetuity present value being determined at time 0, so we save the step of discounting the present value of the perpetuity. The present value at time 0 of the perpetuity-immediate is:

$$1,000a_{\overline{\infty}|9\%} = \frac{1,000}{0.09}$$
$$= \$11,111.11$$

Summing the two time 0 present values, we have:

$$25,670.63 + 11,111.11 = \$36,781.74$$

Solution 2.12

The inflow will be received at times 10 years, 11 years, and so on, with the last payment at time 24 years.

The present value of the outflow at time 7 years is:

$$X$$

The present value of the inflow at time 7 years is:

$$500a_{\overline{15}|4\%}v^2$$

Using the equation of value, we have:

$$X = 500a_{\overline{15}|4\%}v^2 = 500\frac{1-1.04^{-15}}{0.04}(1.04)^{-2} = \$5,139.79$$

Solution 2.13

The inflow comes in at times $n+1$ years, $n+2$ years, and so on, with the last payment at time $2n$ years.

The present value of the outflow at time n years is:

$$4,000$$

The present value of the income at time n years is:

$$438.52a_{\overline{n}|8\%}$$

Using the equation of value we have:

$$4,000 = 438.52a_{\overline{n}|8\%}$$

$$\Rightarrow \frac{1-v^n}{0.08} = \frac{4,000}{438.52} \quad \Rightarrow \quad v^n = 0.2703 \quad \Rightarrow \quad n = \frac{\ln 0.2703}{\ln 1.08^{-1}} = 17$$

Solution 2.14

The present value of the outflow at time 10 years is:

$$1,500$$

The present value of the inflow at time 10 years is:

$$1,000v + 1,000v^2$$

Using the equation of value, we have:

$$1,500 = 1,000v + 1,000v^2 \quad \Rightarrow \quad 2v^2 + 2v - 3 = 0$$

This is a quadratic equation in v so it can be solved using the quadratic formula $v = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$v = \frac{-2 \pm \sqrt{2^2 - (4)(2)(-3)}}{(2)(2)}$$

This gives 0.8229 or -1.8229. The only valid solution is $v = 0.8229$. This gives an interest rate of:

$$i = \frac{1}{v} - 1 = 21.5\%$$

Solution 2.15

The present value of the inflow at time 0 is:

$$400a_{\overline{7|}i} + 1{,}000v^8$$

The present value of the outflow at time 0 is:

$$2{,}500$$

Equating these two:

$$2{,}500 = 400a_{\overline{7|}i} + 1{,}000v^8 = 400\frac{1-(1+i)^{-7}}{i} + 1{,}000(1+i)^{-8}$$

At this point, it is convenient and quick to use trial and error combined with linear interpolation to determine the answer. If this were a multiple-choice exam, we would start with one of the answer choices as our first guess. Let's start with 10% as our first guess.

$$\begin{aligned}400\frac{1-(1.10)^{-7}}{0.10} + 1{,}000(1.10)^{-8} &= 400(4.868419) + 1{,}000(0.466507)\\ &= \$2{,}413.87\end{aligned}$$

Since this is less than the $2,500 that we need for equality in the equation of value, we should decrease the interest rate a little bit. Let's try 9% next.

$$\begin{aligned}400\frac{1-(1.09)^{-7}}{0.09} + 1{,}000(1.09)^{-8} &= 400(5.032953) + 1{,}000(0.501866)\\ &= \$2{,}515.05\end{aligned}$$

We can estimate an unknown interest rate using linear interpolation once we have determined values on each side of the unknown value. If the present values calculated at interest rates i_1 and i_2 are P_1 and P_2 respectively, then the interest rate i corresponding to a present value of P can be approximated by:

$$i \approx i_1 + \frac{P_1 - P}{P_1 - P_2} \times (i_2 - i_1)$$

where $i_1 < i < i_2$ and $P_2 < P < P_1$.

Interpolating between 9% and 10% gives:

$$i = 9.0 + \frac{2{,}515.05 - 2{,}500.00}{2{,}515.05 - 2{,}413.87} \times (10.0 - 9.0) = 9.15\%$$

The equation of value is satisfied with an interest rate of 9.15%.

Solution 2.16

We first convert the problem into one where the first payment takes place at the start or end of the first period so we need to know the accumulated value of $320.74 when the payments start in 5 years:

$$320.74(1.056)^5 = 421.1848$$

This accumulated value is used as the present value of the n-year annuity-due:

$$421.1848 = 40\ddot{a}_{\overline{n}|5.6\%}$$

We solve this equation for the unknown quantity, n:

$$10.52962 = \frac{1-(1.056)^{-n}}{0.056/1.056}$$

$$0.558389 = 1-(1.056)^{-n}$$

$$0.441611 = 1.056^{-n}$$

$$-0.817326 = -n(0.054488)$$

$$n = 15$$

Solution 2.17

We need to know the accumulated value of $50,000 at a point close to when the payments start. We could accumulate to time 14 years and then use an annuity-immediate or accumulate to time 15 years and then use an annuity-due. We will use the former approach.

$$50,000(1.07)^{14} = 128,926.7075$$

There are $25-15+1=11$ annual payments of $\$X$ which start at time 15 years. The accumulated value that we just calculated is used as the present value for the 11-year annuity-immediate.

We set up the equation of value at time 14 years and solve for the unknown X:

$$128,926.7075 = Xa_{\overline{11}|}$$

$$128,926.7075 = X\left(\frac{1-(1.07)^{-11}}{0.07}\right)$$

$$128,926.7075 = X(7.498674)$$

$$X = \$17,193.27$$

Solution 2.18

Interestingly, we don't need to determine the value of i to answer this question.

The perpetuity-immediate has a present value of X divided by i:

$$\text{Present value of perpetuity } = Xa_{\overline{\infty}|i} = \frac{X}{i}$$

The first n payments, which go to Brian, are worth 40% of the perpetuity:

$$0.40\left(\frac{X}{i}\right) = Xa_{\overline{n}|i}$$

$$0.40\left(\frac{X}{i}\right) = X\frac{1-v^n}{i}$$

$$0.40 = 1 - v^n$$

$$v^n = 0.60$$

The present value of Jeff's payments is K times the present value of the perpetuity. The present value of Jeff's payments can also be written as the present value of a perpetuity beginning in $2n$ years:

$$K\left(\frac{X}{i}\right) = v^{2n}\left(\frac{X}{i}\right)$$

$$K = v^{2n} = (0.6)^2 = 0.36$$

Solution 2.19

A financial calculator is very useful and provides a relatively quick way to answer this problem. Since this is a question from the SOA/CAS, we will use the Texas Instruments BA-35, which is one of their approved calculators, to answer this question. However, any financial calculator should also be able to work this question.

The present values of these three annuities are equal. This gives us our equations of value.

Annuity 1: The present value of the perpetuity is 13.7931:

$$a_{\overline{\infty}|7.25\%} = \frac{1}{0.0725} = 13.7931$$

Annuity 2: Using the BA-35 calculator, we can easily solve the following formula for j by pressing 50 and N, 13.7931 and PV, 1 and PMT, then CPT and %i to get:

$$13.7931 = a_{\overline{50}|j}$$

$$j = 0.07$$

Annuity 3: The calculator makes quick work of solving for n by pressing 13.7931 and PV, 6 and %i, 1 and PMT, then CPT and N to get:

$$13.7931 = a_{\overline{n}|j-0.01}$$

$$13.7931 = a_{\overline{n}|0.06}$$

$$n = 30.17 \quad \Rightarrow 30$$

Since the solution needs to be an integer, the answer is rounded to 30.

Solution 2.20

We need to determine the interest rate. We are given the payments that will accumulate to \$8,000 at the end of $3n$ years.

Let's first split the payments into two parts. The first part is the payment of \$98 at the end of each of the first n years. The accumulated value of this annuity-immediate at time n is:

$$98s_{\overline{n}|i}$$

We need to accumulate this accumulated value another $2n$ years to time $3n$ since the total payments will accumulate to \$8,000 at the end of $3n$ years. The accumulated value of the first annuity-immediate at time $3n$ is:

$$98s_{\overline{n}|i}(1+i)^{2n}$$

The second part is the deposit of \$196 at the end of each of the next $2n$ years after the first n years. The accumulated value of this annuity-immediate at time $3n$ is:

$$196s_{\overline{2n}|i}$$

The formula for the accumulated value of an annuity-immediate is $s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}$

We set up the equation of value by putting these parts back together. We are given that $(1+i)^n = 2.0$. We can now solve the equation of value for the interest rate:

$$8,000 = 98s_{\overline{n}|i}(1+i)^{2n} + 196s_{\overline{2n}|i}$$

$$8,000 = 98\frac{(1+i)^n - 1}{i}(1+i)^{2n} + 196\frac{(1+i)^{2n} - 1}{i}$$

$$8,000 = 98\frac{2-1}{i}(2)^2 + 196\frac{2^2 - 1}{i}$$

$$8,000 = 98\frac{4}{i} + 196\frac{3}{i}$$

$$8,000 = \frac{392 + 588}{i}$$

$$8,000i = 980$$

$$i = 0.1225$$