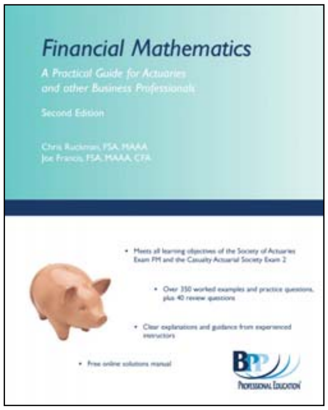

Financial Mathematics
Second Edition

A Practical Guide for Actuaries and other Business Professionals

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Solutions to practice questions – Chapter 4

Solution 4.1

The monthly effective rate of interest is:

$$\frac{i^{(12)}}{12} = 1.09^{\frac{1}{12}} - 1 = 0.007207$$

There are $8 \times 12 = 96$ months in 8 years. Each monthly payment is $4,000/12$. Working in months, the accumulated value is:

$$\frac{4,000}{12}\ddot{s}_{\overline{96|}}$$

This is evaluated at the monthly effective rate of interest. The accumulated value is:

$$\frac{4,000}{12}\ddot{s}_{\overline{96|}0.7207\%} = \frac{4,000}{12} \times \frac{1.007207^{96} - 1}{1 - 1.007207^{-1}} = \$46,236.14$$

Solution 4.2

The nominal interest rate convertible twice per year is $i^{(2)} = 0.08$. The annual effective interest rate is:

$$i = \left(1 + \frac{i^{(2)}}{2}\right)^2 - 1 = \left(1 + \frac{0.08}{2}\right)^2 - 1 = 8.16\%$$

Solution 4.3

Working in years, the present value evaluated at the annual effective rate of interest i is:

$$500a_{\overline{20}|}$$

The nominal interest rate convertible monthly is $i^{(12)} = 0.05$. The annual effective interest rate is:

$$i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1 = \left(1 + \frac{0.05}{12}\right)^{12} - 1 = 5.1162\%$$

The present value is:

$$500a_{\overline{20}|} = 500 \times \frac{1 - 1.051162^{-20}}{0.051162} = \$6,170.17$$

Solution 4.4

The accumulated value evaluated at the annual effective rate of interest i is:

$$35\ddot{s}_{\overline{20}|}$$

The nominal interest rate convertible twice per year is $i^{(2)} = 0.07$. The annual effective interest rate is:

$$i = \left(1 + \frac{i^{(2)}}{2}\right)^{2} - 1 = \left(1 + \frac{0.07}{2}\right)^{2} - 1 = 7.1225\%$$

The accumulated value is:

$$35\ddot{s}_{\overline{20}|} = 35 \times \frac{(1+i)^{20} - 1}{d} = 35 \times \frac{1.071225^{20} - 1}{1 - 1.071225^{-1}} = \$1,557.76$$

Solution 4.5

Given a nominal discount rate convertible 4 times per year, the annual effective interest rate is:

$$\begin{aligned} i &= \left(1 - \frac{d^{(4)}}{4}\right)^{-4} - 1 \\ &= \left(1 - \frac{0.06}{4}\right)^{-4} - 1 \\ &= 6.23\% \end{aligned}$$

Solution 4.6

The present value evaluated at the annual effective rate of interest i is:

$$1{,}000a_{\overline{5}|}$$

The nominal discount rate convertible twice per year is $d^{(2)}=0.1$. The annual effective discount rate is:

$$d=1-\left(1-\frac{d^{(2)}}{2}\right)^2=1-\left(1-\frac{0.1}{2}\right)^2=0.0975$$

This means that:

$$v=1-d=0.9025$$

Since $i=\frac{1}{v}-1$, the present value is:

$$1{,}000a_{\overline{5}|}=1{,}000\times\frac{1-0.9025^5}{0.9025^{-1}-1}=\$3{,}714.26$$

Solution 4.7

The accumulated value evaluated at the annual effective rate of interest i is:

$$455\ddot{s}_{\overline{15}|}$$

The nominal discount rate convertible monthly is $d^{(12)}=0.035$. The annual effective discount rate is:

$$d=1-\left(1-\frac{d^{(12)}}{12}\right)^{12}=1-\left(1-\frac{0.035}{12}\right)^{12}=0.034444$$

This means that:

$$v=1-d=0.965556$$

So the accumulated value is:

$$455\ddot{s}_{\overline{15}|}=455\times\frac{0.965556^{-15}-1}{0.034444}=\$9{,}138.00$$

Solution 4.8

Working in years, the present value evaluated at the annual effective rate of interest i is:

$$4{,}800a^{(12)}_{\overline{7}|}$$

The nominal interest rate convertible 3 times per year is $i^{(3)}=0.09$. The annual effective interest rate is:

$$i=\left(1+\frac{i^{(3)}}{3}\right)^3-1=\left(1+\frac{0.09}{3}\right)^3-1=9.2727\%$$

The nominal interest rate convertible monthly is:

$$i^{(12)}=12\left((1.092727)^{\frac{1}{12}}-1\right)=0.089005$$

So the present value is:

$$4,800a_{\overline{7}|}^{(12)} = 4,800 \times \frac{1-1.092727^{-7}}{0.089005} = \$24,939.80$$

Alternatively, we could also work in months. The monthly effective interest rate is:

$$\frac{i^{(12)}}{12} = 1.092727^{\frac{1}{12}} - 1 = 0.007417$$

There are 84 months in 7 years. The present value is:

$$\frac{4,800}{12} a_{\overline{84}|0.7417\%}$$

The present value is:

$$\frac{4,800}{12} a_{\overline{84}|} = \frac{4,800}{12} \times \frac{1-1.007417^{-84}}{0.007417} = \$24,939.80$$

Solution 4.9

Working in years, the accumulated value evaluated at the annual effective rate of interest i is:

$$6,880 s_{\overline{16}|}^{\left(\frac{1}{2}\right)}$$

The nominal interest rate convertible twice per year is $i^{(2)} = 0.1$. The annual effective interest rate is:

$$i = \left(1 + \frac{i^{(2)}}{2}\right)^2 - 1 = \left(1 + \frac{0.1}{2}\right)^2 = 0.1025$$

The nominal every-other-year interest rate is:

$$i^{\left(\frac{1}{2}\right)} = \frac{1}{2}\left((1.1025)^2 - 1\right) = 0.107753$$

So the accumulated value is:

$$6,880 s_{\overline{16}|}^{\left(\frac{1}{2}\right)} = 6,880 \times \frac{1.1025^{16} - 1}{0.107753} = \$240,390.22$$

Alternatively, working in two-yearly periods, the two-yearly effective rate of interest is:

$$\frac{i^{\left(\frac{1}{2}\right)}}{\frac{1}{2}} = 1.1025^2 - 1 = 21.5506\%$$

The accumulated value is:

$$2 \times 6,880 s_{\overline{8}|21.5506\%} = 13,760 \times \frac{1.215506^8 - 1}{0.215506} = \$240,390.22$$

Solution 4.10

Working in years, the present value evaluated at the annual effective rate of interest i is:

$$12 \times 50\ddot{a}_{\overline{7}|}^{(12)}$$

The nominal interest rate convertible quarterly is $i^{(4)} = 0.04$. The annual effective interest rate is:

$$i = \left(1 + \frac{i^{(4)}}{4}\right)^4 - 1 = \left(1 + \frac{0.04}{4}\right)^4 - 1 = 0.040604$$

The annual effective discount rate is:

$$d = \frac{0.040604}{1.040604} = 0.039020$$

The nominal discount rate convertible monthly is:

$$d^{(12)} = 12\left(1 - (1 - 0.039020)^{\frac{1}{12}}\right) = 0.039735$$

So the present value is:

$$12 \times 50\ddot{a}_{\overline{7}|}^{(12)} = 12 \times 50 \times \frac{1 - 1.040604^{-7}}{0.039735} = \$3,671.76$$

Solution 4.11

Working in years, the accumulated value evaluated at the annual effective rate of interest i is:

$$50,000\ddot{s}_{\overline{20}|}^{(2)}$$

The nominal interest rate convertible monthly is $i^{(12)} = 0.06$. The annual effective interest rate is:

$$i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1 = 0.061678$$

The annual effective discount rate is:

$$d = \frac{0.061678}{1.061678} = 0.058095$$

From this the nominal discount rate convertible monthly $d^{(2)}$ is:

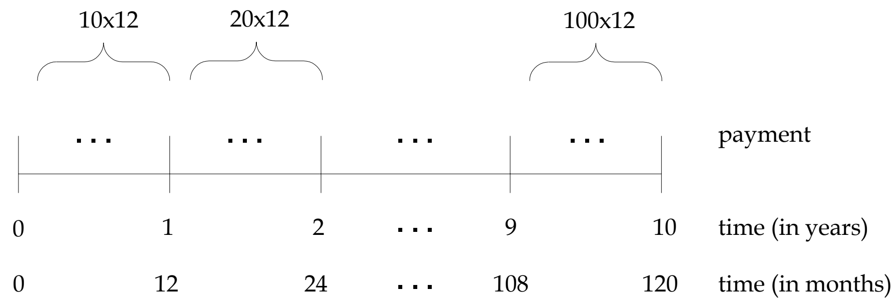
$$d^{(2)} = 2\left(1 - (1 - 0.05895)^{\frac{1}{2}}\right) = 0.058964$$

So the accumulated value is:

$$50,000\ddot{s}_{\overline{20}|}^{(2)} = 50,000 \times \frac{1.061678^{20} - 1}{0.058964} = \$1,959,000.90$$

Solution 4.12

Let's first draw the timeline diagram:



The factor $(Ia)_{\overline{10|}}^{(12)}$ expects a payment of $1/12$ at the end of each month during the first year, $2/12$ at the end of each month during the second year, and so on. This payment series is 120 times that. So, the present value is:

$$120(Ia)_{\overline{10|}}^{(12)} = 120 \times \frac{\ddot{a}_{\overline{10|}} - 10v^{10}}{i^{(12)}}$$

Calculating the required values:

$$\frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$$

$$i = \left(1 + \frac{0.12}{12}\right)^{12} - 1 = 12.6825\%$$

$$\ddot{a}_{\overline{10|}} = \frac{1-(1.126825)^{-10}}{0.126825/1.126825} = 6.192807$$

$$(Ia)_{\overline{10|}}^{(12)} = \frac{6.192807 - 10(1.126825)^{-10}}{0.12} = 26.357160$$

The present value of the payments is:

$$120(Ia)_{\overline{10|}}^{(12)} = 120 \times 26.357160 = \$3,162.86$$

Solution 4.13

The accumulated value at 15 years is:

$$4 \times 35 (I\ddot{s})^{(4)}_{\overline{15|}} = 4 \times 35 \times \frac{\ddot{s}_{\overline{15|}} - 15}{d^{(4)}}$$

Calculating the required values:

$$i = \left(1 + \frac{0.12}{12}\right)^{12} - 1 = 12.6825\%$$

$$d^{(4)} = 4\left[1 - (1.126825)^{-\frac{1}{4}}\right] = 11.7639\%$$

$$\ddot{s}_{\overline{15|}} = \frac{1.126825^{15} - 1}{0.126825/1.126825} = 44.387095$$

$$(I\ddot{s})^{(4)}_{\overline{15|}} = \frac{44.387095 - 15}{0.117639} = 249.806556$$

The present value of the payments is:

$$4 \times 35 (I\ddot{s})^{(4)}_{\overline{15|}} = 4 \times 35 \times 249.806556 = \$34,972.92$$

Solution 4.14

Here, $p = 2$. Working in years, the present value of payments of $1/4$ now, $2/4$ after 6 months, $3/4$ after 12 months, and so on, is:

$$(I^{(2)}\ddot{a})^{(2)}_{\overline{6|}} = \frac{\ddot{a}^{(2)}_{\overline{6|}} - 6v^6}{d^{(2)}}$$

Since we have payments of 5, 10, 15, and so on, the present value is:

$$5 \times 2^2 \times (I^{(2)}\ddot{a})^{(2)}_{\overline{6|}} = 20 \times \frac{\ddot{a}^{(2)}_{\overline{6|}} - 6v^6}{d^{(2)}}$$

Calculating the required values when the nominal discount rate convertible every 6 months is 12%:

$$i = \left(1 - \frac{d^{(2)}}{2}\right)^{-2} - 1 = \left(1 - \frac{0.12}{2}\right)^{-2} - 1 = 13.1734\%$$

$$a_{\overline{6|}} = \frac{1 - 1.131734^{-6}}{0.131734} = 3.978323$$

$$\ddot{a}^{(2)}_{\overline{6|}} = \frac{0.131734}{0.12}(3.978323) = 4.367331$$

So the present value is:

$$20\left(\frac{4.367331-6(1.131734)^{-6}}{0.12}\right)=\$251.97$$

Alternatively, working in semiannual periods, the semiannual effective interest rate is:

$$\frac{i^{(2)}}{2}=1.131734^{\frac{1}{2}}-1=6.3830\%$$

The factor $(I\ddot{a})_{\overline{12}|}$ expects a payment of 1 at the end of the first period, 2 at the end of the second period, and so on. This payment series is 5 times that. So, the present value is:

$$5(I\ddot{a})_{\overline{6\times 2}|6.3830\%}=5\times\frac{\ddot{a}_{\overline{12}|6.3830\%}-12v^{12}}{d^{(2)}/2}$$

Calculating the required values:

$$\ddot{a}_{\overline{12}|6.3830\%}=\frac{1-1.063830^{-12}}{0.063830/1.063830}=8.734661$$

The present value is:

$$5\times\frac{8.734661-12(1.063830)^{-12}}{0.12/2}=\$251.97$$

Solution 4.15

There are 20 quarters in 5 years. The monthly increasing payments can be split into a level series of monthly payments and an increasing series of monthly payments. The timeline diagram is:

	50	55	60	. . .	145	payment
0	1	2	3	. . .	20	time (in quarters)
	5	10	15	. . .	100	increasing payment
	45	45	45	. . .	45	level payment

Working in years, the accumulated value at 5 years is:

$$4\times 45s_{\overline{5}|}^{(4)}+5\times 4^2\times(I^{(4)}s)_{\overline{5}|}^{(4)}=180s_{\overline{5}|}^{(4)}+80\times\frac{\ddot{s}_{\overline{5}|}^{(4)}-5}{d^{(4)}}$$

Calculating the required values when the annual effective interest rate is 6%:

$$i^{(4)} = 4\left[1.06^{\frac{1}{4}} - 1\right] = 5.8695\%$$

$$d^{(4)} = 4\left[1 - 1.06^{-\frac{1}{4}}\right] = 5.7847\%$$

$$s_{\overline{5|}} = \frac{1.06^5 - 1}{0.06} = 5.637093$$

$$s_{\overline{5|}}^{(4)} = \frac{0.06}{0.058695}(5.637093) = 5.762388$$

$$\ddot{s}_{\overline{5|}}^{(4)} = \frac{0.06}{0.057847}(5.637093) = 5.846944$$

So the accumulated value is:

$$180 \times 5.762388 + 80\left(\frac{5.846944 - 5}{0.058695}\right) = \$2,191.59$$

Alternatively, working in quarters, the quarterly effective interest rate is:

$$\frac{i^{(4)}}{4} = \frac{0.058695}{4} = 1.4674\%$$

The accumulated value of the payments of 5, 10, 15 and so on is:

$$5(Is)_{\overline{4\times5|}1.4674\%} = 5 \times \frac{\ddot{s}_{\overline{20|}1.4674\%} - 20}{0.014674}$$

Calculating the required values:

$$s_{\overline{20|}1.4674\%} = \frac{1.014674^{20} - 1}{0.014674} = 23.049552$$

$$\ddot{s}_{\overline{20|}1.4674\%} = 23.049552(1.014674) = 23.387777$$

The accumulated value of the level quarterly payment of \$45 and the quarterly increasing annuity-immediate is:

$$45 \times 23.049552 + 5\left(\frac{23.387777 - 20}{0.014674}\right) = \$2,191.59$$

Solution 4.16

Here, $p=6$. The present value of payments of $1/36$ after 2 months, $2/36$ after 4 months, $3/36$ after 6 months, and so on, is:

$$(I^{(6)}a)_{\overline{10}|}^{(6)}=\frac{\ddot{a}_{\overline{10}|}^{(6)}-10v^{10}}{i^{(6)}}$$

Since we have payments of 5, 10, 15, and so on, the present value is:

$$5\times 36(I^{(6)}a)_{\overline{10}|}^{(6)}=180\frac{\ddot{a}_{\overline{10}|}^{(6)}-10v^{10}}{i^{(6)}}$$

Calculating the required values when $i=0.08$:

$$i^{(6)}=6\left(1.08^{\frac{1}{6}}-1\right)=0.077457$$

$$d^{(6)}=6\left(1-1.08^{-\frac{1}{6}}\right)=0.076470$$

$$\ddot{a}_{\overline{10}|}^{(6)}=\frac{1-1.08^{-10}}{0.076470}=7.019872$$

So the present value X is:

$$X=180\left(\frac{7.019872-10(1.08)^{-10}}{0.077457}\right)=\$5,549.27$$

Solution 4.17

We recall that the present value of a perpetuity-immediate is just the level periodic payment divided by the effective periodic interest rate. To simplify the notation, let's define j as the effective interest rate for a three-year period:

$$j=(1+i)^3-1$$

In three years, the perpetuity will have a present value of 10 divided by j. To find the present value at time zero, this present value must be discounted for three years, and this is accomplished by dividing by $(1+j)$. The equation of value becomes:

$$\frac{10}{j}\left(\frac{1}{1+j}\right)=32$$

$$10=32j^2+32j$$

$$3.2j^2+3.2j-1=0$$

Using the quadratic formula:

$$j=\frac{-3.2\pm\sqrt{(3.2)^2+4(3.2)(1.0)}}{2(3.2)}=0.25 \quad \text{(since we are only interested in positive values)}$$

Now we can find i , the annual effective interest rate:

$$0.25 = (1+i)^3 - 1$$

$$i = (1.25)^{\frac{1}{3}} - 1 = 0.077217$$

The present value of a perpetuity-immediate that pays \$1 at the end of each 4-month period is just \$1 divided by the effective interest rate for 4 months (one third of a year):

$$X = \frac{1}{(1.077217)^{\frac{1}{3}} - 1} = \$39.83$$

Solution 4.18

With a constant rate of compound interest, the force of interest is constant. With a constant rate of simple interest, the force of interest is not constant.

The force of interest is constant for Tawny:

$$e^{\delta} = \left(1 + \frac{0.10}{2}\right)^2$$

$$\delta = 2\ln(1.05) = 0.097580$$

The force of interest for Fabio depends on when it is measured. Let i be the simple interest rate earned by Fabio. The accumulated value under simple interest at time t is $1+ti$. We recall that the force of interest at time t is the derivative of the accumulated value with respect to t divided by the accumulated value at time t :

$$\delta_t = \frac{AV_t^{'}}{AV_t} = \frac{i}{1+ti}$$

$$\delta_5 = \frac{i}{1+5i}$$

Since the force of interest on the two accounts is equal at the end of 5 years, we can find the simple interest rate earned by Fabio:

$$0.097580 = \frac{i}{1+5i}$$

$$0.097580(1+5i) = i$$

$$0.097580 + 0.487902i = i$$

$$i = 0.190550$$

At the end of 5 years, Fabio has \$1,952.75:

$$1,000\left[1+5(0.190550)\right] = \$1,952.75$$

Solution 4.19

This 5-year increasing annuity-immediate pays 2 at the end of the first month, 4 at the end of the second month, 6 at the end of the third month, and each month thereafter the payment increases by 2 until the final payment of 120 at the end of the 60th month. We are also given a nominal interest rate of 9% convertible quarterly, but we need the monthly effective interest rate.

To calculate a monthly effective interest rate from a quarterly nominal rate $i^{(4)}$, we first compute the quarterly effective rate $\frac{i^{(4)}}{4}$, then we calculate the annual effective rate i, and then we can calculate the monthly effective interest rate $\frac{i^{(12)}}{12}$:

$$\frac{i^{(4)}}{4}=\frac{0.09}{4}=0.0225$$

$$i=\left(1+\frac{i^{(4)}}{4}\right)^4-1=(1.0225)^4-1=0.093083$$

$$\frac{i^{(12)}}{12}=(1+i)^{\frac{1}{12}}-1=(1.0903083)^{\frac{1}{12}}-1=0.007444$$

Alternatively, the monthly effective interest rate can be calculated directly from the quarterly effective interest rate:

$$\frac{i^{(12)}}{12}=\left(1+\frac{i^{(4)}}{4}\right)^{\frac{1}{3}}-1=(1.0225)^{\frac{1}{3}}-1=0.007444$$

We can now set up the equation of value for the present value of the increasing annuity-immediate:

$$PV=X=\frac{2}{1.007444}+\frac{4}{(1.007444)^2}+\frac{6}{(1.007444)^3}+\cdots+\frac{120}{(1.007444)^{60}}$$

$$=2\left[\frac{1}{1.007444}+\frac{2}{(1.007444)^2}+\frac{3}{(1.007444)^3}+\cdots+\frac{60}{(1.007444)^{60}}\right]$$

The part in the brackets is an increasing annuity-immediate. It can be calculated by:

$$X=2\left[(Ia)_{\overline{60}|0.7444\%}\right]$$

$$=2\left[\frac{\ddot{a}_{\overline{60}|0.7444\%}-60v^{60}_{0.7444\%}}{0.007444}\right]$$

Calculating the required value:

$$\ddot{a}_{\overline{60}|0.7444\%}=\frac{1-(1.007444)^{-60}}{0.007444/1.007444}=48.607728$$

The present value is:

$$X=2\left[\frac{48.607728-60(1.007444)^{-60}}{0.007444}\right]$$

$$=\$2{,}729.21$$

Solution 4.20

The accumulated value after 1 year is:

$$100\left(1-\frac{d^{(4)}}{4}\right)^{-4}=100\left(1-\frac{0.075}{4}\right)^{-4}$$
$$=107.865192$$

This accumulated value is accumulated for another year to determine the accumulated value after 2 years:

$$107.865192\left(1+\frac{i^{(4)}}{4}\right)^{4}=107.865192\left(1+\frac{0.075}{4}\right)^{4}$$
$$=\$116.19$$

