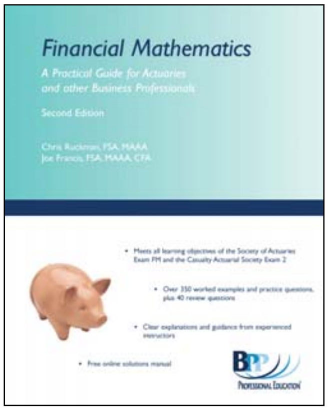

Financial Mathematics
Second Edition
A Practical Guide for Actuaries and other Business Professionals

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Solutions to practice questions – Chapter 5

Solution 5.1

The net present value is:

$$4 \times 620 a_{\overline{5}|}^{(4)} - 5,000$$

evaluated using $i^{(4)} = 4\%$.

First we need to find i:

$$i = \left(1 + \frac{i^{(4)}}{4}\right)^4 - 1 = \left(1 + \frac{0.04}{4}\right)^4 - 1 \quad \Rightarrow \quad i = 4.0604\%$$

So the net present value is:

$$4 \times 620 \frac{1 - 1.040604^{-5}}{0.04} - 5,000 = \$6,188.24$$

We could also work in quarters using a 1% quarterly effective rate of interest. The net present value is then:

$$620 a_{\overline{20}|1\%} - 5,000 = \$6,188.24$$

Solution 5.2

The project's NPV is:

$$\text{NPV} = \frac{-10,000}{(1.11)^0} + \frac{10,000}{(1.11)^1} + \frac{2,000}{(1.11)^2} = \$632.25$$

We set up the equation of value that equated the NPV to zero and solve for the IRR:

$$\text{NPV} = \sum \frac{\text{net CF}_t}{(1+IRR)^t} = 0$$

$$0 = -10,000 + \frac{10,000}{(1+IRR)} + \frac{2,000}{(1+IRR)^2}$$

If we let $x = 1 + IRR$, then we can re-arrange the formula to $10,000x^2 - 10,000x - 2,000 = 0$, which we can solve using the quadratic equation. After doing this, we get $x = 1.1708$, or $IRR = 17.08\%$.

Solution 5.3

The IRR for this 5-year bond is:

$$IRR = \left(\frac{100}{85}\right)^{\frac{1}{5}} - 1 = 3.3038\%$$

Solution 5.4

The formula for calculating the IRR is:

$$\text{NPV} = \sum \frac{\text{net CF}_t}{(1+IRR)^t} = 0$$

$$0 = -50,000 + \frac{15,000}{(1+IRR)^1} + \frac{40,000}{(1+IRR)^2} + \frac{10,000}{(1+IRR)^3}$$

It is difficult to solve the above formula for the IRR without a spreadsheet. If we didn't have the fourth term of $10,000 at the end of the third year in the above equation, we could have solved for the IRR using the quadratic equation. But we do have the fourth term. You might want to try using a spreadsheet to solve for the IRR to see how the formula works. Without a spreadsheet, try plugging a solution into the formula using trial and error to narrow down the options until the answer is found. In the above example, if we start with an IRR guess of 10%, we get a value of:

$$-50,000 + \frac{15,000}{(1.1)^1} + \frac{40,000}{(1.1)^2} + \frac{10,000}{(1.1)^3} = 4,207.36$$

Since we want the equation to equal zero, we need a higher IRR. If our second guess for the IRR is 15%, the equation is equal to –135.61, which is a lot closer to zero. After several guesses, we get an IRR of 14.83% that makes the equation approximately equal to zero.

Solution 5.5

At the end of the first and subsequent years, the first investment of $30,000 will receive $0.04 \times \$30,000 = \$1,200$ interest. Similarly at the end of the second and subsequent years, $1,200 interest will be earned on the second investment of $30,000.

So the value of the investment is:

$$20 \times 30,000 + 1,200(1.02)^{19} + 2 \times 1,200(1.02)^{18} + \cdots + 20 \times 1,200$$
$$= 20 \times 30,000 + 1,200(Is)_{\overline{20|}}$$

where the increasing annuity function is evaluated at 2%.

So the value of the investment is:

$$20\times 30{,}000+1{,}200(1.02)^{20}\,\frac{\ddot{a}_{\overline{20|}}-20v^{20}}{0.02}=\$886{,}999.03$$

Solution 5.6

The nominal interest rate used to accumulate the nominal cash flows is:

$$i=(1.05)(1.03)-1=8.15\%$$

The accumulated value of the cash flows is:

$$100(1.0815)^5+105(1.0815)^4+110(1.0815)^3+115(1.0815)^2+120(1.0815)+125$$

This can be split into a 6-year level annuity-immediate of \$95 and a 6-year increasing annuity-immediate where the first payment is \$5. The accumulated value can be rewritten as:

$$95s_{\overline{6|}8.15\%}+5(Is)_{\overline{6|}8.15\%}$$

Calculating the required values, we have:

$$s_{\overline{6|}8.15\%}=\frac{(1.0815)^6-1}{0.0815}=7.363733$$
$$\ddot{s}_{\overline{6|}8.15\%}=(1.0815)(7.363733)=7.963878$$
$$(Is)_{\overline{6|}8.15\%}=\frac{7.963878-6}{0.0815}=24.096659$$

The accumulated value is then:

$$95\times 7.363733+5\times 24.096659=\$820.04$$

Solution 5.7

Each spouse will have purchased 1,000 shares at the start of the year.

On June 30, Mrs. Rich will have purchased an additional 800 shares (at a price of \$1.25 each), making a total of 1,800 shares, which will be worth \$1,800 at the year end.

Mr. Rich, however, will have purchased an additional 1,250 shares (at a price of \$0.80), making a total of 2,250 shares, which will be worth \$2,250.

The equation of value for Mrs. Rich is:

$$1{,}000(1+i)+1{,}000(1+i)^{0.5}=1{,}800$$

Solving this using the quadratic formula gives a dollar-weighted rate of interest of -13.2%.

For Mr. Rich, we have:

$$1{,}000(1+i)+1{,}000(1+i)^{0.5}=2{,}250$$

The dollar-weighted rate of interest is $+16.9\%$.

Solution 5.8

Drawing up a table of the values of the fund just before the cash flows and the cash flows we have:

Date	Value of the fund just before the cash flows	Cash flows
1/1/03	4.0	
6/30/03	4.3	1
3/31/04	5.2	1.5
11/1/04	6.0	–4.57
12/31/04	2.0	

The equation for the time-weighted rate of interest is:

$$(1+i)^2 = \frac{4.3}{4} \times \frac{5.2}{4.3+1} \times \frac{6}{5.2+1.5} \times \frac{2}{6-4.57} = 1.321$$

This means that the time-weighted rate of interest is:

$$i = (1.321)^{0.5} - 1 = 14.9\%$$

Solution 5.9

The \$100 deposited at the beginning of 1995 earns the new money rate of t in 1995, 5.5% in 1996, the portfolio rate of 4.5% in 1997 and the portfolio rate of 5.0% in 1998. The interest earned in 1998 from this \$100 investment made at the beginning of 1995 is:

$$100(1+t)(1.055)(1.045)(0.05) = 5.51 + 5.51t$$

The \$100 deposited at the beginning of 1997 earns the new money rate of 7% in 1997 and the new money rate of $(t+2.5\%)$ in 1998. The interest earned in 1998 from this \$100 investment made at the beginning of 1997 is:

$$100(1.07)(t+0.025) = 107t + 2.675$$

Together, these two amounts sum to 13.81. We set up the equation and solve for t:

$$\begin{aligned} 13.81 &= 5.51 + 5.51t + 107t + 2.675 \\ 112.51t &= 5.625 \\ t &= 0.05 \end{aligned}$$

Solution 5.10

Let the quarterly payment be X. Using the prospective method, the equation of value, working in years, is:

$$10{,}000 = 4Xa_{\overline{10}|}^{(4)}$$

Solving this we determine the quarterly payment:

$$10{,}000 = 4X \frac{1-1.07^{-10}}{4\left((1.07)^{\frac{1}{4}} - 1\right)}$$

$$\Rightarrow \quad X = \frac{10{,}000}{4 \times 7.205352} = \$346.96$$

Solution 5.11

The payment is $346.96 from the previous solution. After the sixth payment there are 8.5 years left to go. Using the prospective method, the principal outstanding is:

$$4 \times 346.96 a_{\overline{8.5}|}^{(4)} = 4 \times 346.96 \times \frac{1 - 1.07^{-8.5}}{4\left((1.07)^{\frac{1}{4}} - 1\right)} = 8,895.44$$

So the principal outstanding is $8,895.44.

Solution 5.12

The total amount of interest is the total of the payments less the amount of the loan.

Here there are 40 quarterly payments. So the total amount of interest is:

$$40 \times 346.96 - 10,000 = 3,878.40$$

So the total amount of interest paid is $3,878.40.

Solution 5.13

The principal outstanding after the 9th payment, when the loan has 7.75 years left to go, is:

$$B_9 = 4 \times 346.96 a_{\overline{7.75}|}^{(4)} = 4 \times 346.96 \times \frac{1 - 1.07^{-7.75}}{4\left((1.07)^{\frac{1}{4}} - 1\right)} = 8,299.75$$

Since payments are made quarterly, the interest due is calculated by applying the effective quarterly interest rate to the amount outstanding. So the interest paid in the tenth payment is:

$$I_{10} = \left((1.07)^{\frac{1}{4}} - 1\right) 8,299.75 = \$141.58$$

The principal paid in the tenth payment is:

$$P_{10} = 346.96 - 141.58 = \$205.38$$

Solution 5.14

The 10-year annuity-immediate factor at 10% is:

$$a_{\overline{10}|10\%} = \frac{1 - (1.10)^{-10}}{0.10} = 6.144567$$

The initial loan payment is:

$$X = \frac{50,000}{a_{\overline{10}|10\%}} = \frac{50,000}{6.144567} = \$8,137.27$$

The outstanding loan balance after 3 years by the prospective method is:

$$B_3 = 8,137.27 a_{\overline{7}|10\%} = 8,137.27 \times \frac{1 - (1.10)^{-7}}{0.10} = \$39,615.64$$

At this time, the loan is refinanced for 7 more years at an interest rate of 6%. The 7-year annuity-immediate factor at 6% is:

$$a_{\overline{7}|6\%} = \frac{1-(1.06)^{-7}}{0.06} = 5.582381$$

The new loan payment after refinancing is:

$$X = \frac{39,615.64}{5.582381} = \$7,096.55$$

Solution 5.15

Part (i)

If the first payment is X, then the equation of value is:

$$\begin{aligned} 80,000 &= Xv + (X+1,600)v^2 + (X+2\times 1,600)v^3 + \cdots + (X+9\times 1,600)v^{10} \\ &= (X-1,600)a_{\overline{10}|} + 1,600(Ia)_{\overline{10}|} \end{aligned}$$

Using an annual effective rate of interest of 5%, we have:

$$\ddot{a}_{\overline{10}|} = \frac{1-1.05^{-10}}{1-1.05^{-1}} = 8.107822$$

$$a_{\overline{10}|} = \frac{1-1.05^{-10}}{0.05} = 7.721735$$

$$(Ia)_{\overline{10}|} = \frac{8.107822-(10)(1.05)^{-10}}{0.05} = 39.373783$$

Substituting these values into the equation of value, we get $X = \$3,801.83$.

Part (ii)

The interest due in the first payment period is $I_1 = 0.05\times 80,000 = \$4,000$. Notice that this is greater than the amount of the first loan payment. In this case, the first loan payment is not enough to cover the amount of interest due in the first period. The lender requires $4,000 of interest due on the $80,000 loan over the first year, so the difference between $4,000.00 and $3,801.83 is added to the outstanding balance at the end of the first year since the borrower still owes that amount to the lender. The amortization schedule for this loan is provided below to illustrate how the outstanding balance becomes zero at time ten years when the loan is scheduled to be paid off. In the table headings, BOY = beginning of year and EOY = end of year.

Time	BOY Balance	Payment	Interest due	EOY Balance
1	$ 80,000.00	**$ 3,801.83**	**$ 4,000.00**	$ 80,198.17
2	$ 80,198.17	$ 5,401.83	$ 4,009.91	$ 78,806.25
3	$ 78,806.25	$ 7,001.83	$ 3,940.31	$ 75,744.73
4	$ 75,744.73	$ 8,601.83	$ 3,787.24	$ 70,930.14
5	$ 70,930.14	$ 10,201.83	$ 3,546.51	$ 64,274.81
6	$ 64,274.81	$ 11,801.83	$ 3,213.74	$ 55,686.73
7	$ 55,686.73	$ 13,401.83	$ 2,784.34	$ 45,069.23
8	$ 45,069.23	$ 15,001.83	$ 2,253.46	$ 32,320.86
9	$ 32,320.86	$ 16,601.83	$ 1,616.04	$ 17,335.08
10	$ 17,335.08	$ 18,201.83	$ 866.75	$ (0.00)

Solution 5.16

For the amortization method, let the annual payment be X , then:

$$50,000 = Xa_{\overline{15}|6\%}$$

This can be used to find X :

$$X = \frac{50,000}{9.712249} = 5,148.14$$

For the sinking fund method, if we have a payment into the sinking fund of Y then:

$$Ys_{\overline{15}|5.5\%} = 50,000$$

This can be used to find Y :

$$Y = \frac{50,000}{22.408664} = 2,231.28$$

The total payments are the same, so $50,000 \times i + 2,231.28 = 5,148.14$.

This means that $i = 5.83\%$.

Solution 5.17

We are given that two projects have the same internal rate of return (IRR). Project X requires an initial investment of $5,000 and generates net cash flows of $300 at the end of every 6 months forever. Using this information, we can solve for project X's IRR. We then use this IRR to solve for the single cash flow Z that is generated by project Y. Project Y also requires an initial investment of $5,000. Once we know Z, we can determine project Y's NPV.

Project X generates a perpetuity-immediate of 300 at the end of every 6 months. The formula for the present value of a perpetuity-immediate that pays $1 at the end of every year is:

$$a_{\overline{\infty}|i} = \frac{1}{i}$$

To modify the formula for payments at the end of every 6 months, we need to use an effective semiannual interest rate:

$$a_{\overline{\infty}|} = \frac{1}{\frac{i^{(2)}}{2}} = \frac{1}{(1+i)^{0.5} - 1}$$

The IRR is that return which causes the NPV to equal zero. The NPV for project X is:

$$NPV_X = -5,000 + \frac{300}{(1+i)^{0.5} - 1}$$

$$0 = -5,000 + \frac{300}{(1+IRR)^{0.5} - 1}$$

$$5,000 = \frac{300}{(1+IRR)^{0.5} - 1}$$

$$\frac{300}{5,000} = (1+IRR)^{0.5} - 1$$

$$1.06 = (1+IRR)^{0.5}$$

$$1 + IRR = 1.1236$$

$$IRR = 0.1236$$

We are given that the IRR is the same for both projects. Project Y requires an initial investment of 5,000 and generates just one cash flow of Z at 5 years. We set up the NPV formula, set it equal to zero, and solve for the IRR of project Y:

$$NPV_Y = -5,000 + \frac{Z}{(1+i)^5}$$

$$0 = -5,000 + \frac{Z}{(1+IRR)^5}$$

$$5,000 = \frac{Z}{(1+0.1236)^5}$$

$$Z = 5,000(1.1236)^5 = 5,000(1.79085)$$

$$Z = 8,954.24$$

Using an annual effective interest rate of 10%, the NPV of project Y is:

$$NPV_Y = -5,000 + \frac{8,954.24}{(1.10)^5} = -5,000 + 5,559.88 = \$559.88$$

Solution 5.18

The amortization method is the standard compound interest method in which level payments are used to pay off a loan. The sinking fund method calls for the borrower to pay the interest on the loan each year and make a level payment to a sinking fund that will grow at interest rate j. The sinking fund must grow to equal the full amount of the principal to pay off the loan when the loan matures.

We can use the information provided about the amortization method to find the value of X:

$$20,000 = Xa_{\overline{20}|0.065}$$

$$X = \frac{20,000}{11.018507} = \$1,815.1279$$

In the sinking fund method, a portion of the \$1,815.1279 is used to make level interest payments, and the rest is accrued at an interest rate of j:

$$20,000 = [1,815.1279 - 0.08(20,000)]s_{\overline{20}|j}$$

$$s_{\overline{20}|j} = 92.9679$$

Using a financial calculator is probably the quickest way to solve for j. Without a financial calculator, we can determine j by trial and error using:

$$\frac{(1+j)^{20} - 1}{j} = 92.9679$$

We have:

$$j = 14.18\%$$

Solution 5.19

The time-weighted interest rate is the value of i that satisfies:

$$(1+i)^T = \left(\frac{F_1}{F_0}\right)\left(\frac{F_2}{F_1+c_1}\right)\left(\frac{F_3}{F_2+c_2}\right)\cdots\left(\frac{F_T}{F_n+c_n}\right)$$

where F_0 is the initial fund value, F_T is the final fund value, $c_1,\cdots,c_n$ are the external cash flows, and $F_1,\cdots,F_n$ are the values of the fund just before those cash flows. For investment account L, we have an initial fund value of 100.0, a fund value of 125.0 at time $t=0.5$ just before the withdrawal of X, and a final fund value at time $t=1$ of 105.8. At the time of 1 year, we have:

$$(1+i)^1 = \left(\frac{125}{100}\right)\left(\frac{105.8}{125-X}\right) = \frac{132.25}{125-X}$$

The dollar-weighted rate of return is the interest rate that satisfies this equation: the final fund value equals the accumulated value of the initial fund plus the accumulated value of the cash flows:

$$F_T = F_0(1+i)^T + c_1(1+i)^{T-t_1} + c_2(1+i)^{T-t_2} + \cdots + c_n(1+i)^{T-t_n}$$

For investment account K, we have an initial fund value of 100.0, a fund value of 125.0 at time $t=0.5$ just before the withdrawal of X, a fund value of 110.0 at time $t=0.75$ just before the deposit of $2X$, and a final fund value at time $t=1$ of 125.0. We notice that the intermediate fund values are not needed for the dollar-weighted return calculation.

At the time of 1 year, we have:

$$F_1 = 125 = 100(1+i)^1 + (-X)(1+i)^{1-0.5} + 2X(1+i)^{1-0.75}$$
$$125 = 100(1+i) - X(1+i)^{0.5} + 2X(1+i)^{0.25}$$

Since the time period involved is one year, we expand the equation using the first-order binomial expansion, that is, we replace $(1+i)^n$ with $(1+ni)$ as an approximation to make it easier to find the dollar-weighted interest rate:

$$125 = 100(1+i) - X(1+0.5i) + 2X(1+0.25i)$$
$$125 = 100 + 100i - X - 0.5Xi + 2X + 0.5Xi$$
$$25 = X + 100i$$
$$i = \frac{25-X}{100}$$

From the time-weighted account L result, we have an expression for $1+i$. We use this with the dollar-weighted account K result and solve for the interest rate:

$$1+i = \frac{132.25}{125-X}$$
$$1+\frac{25-X}{100} = \frac{132.25}{125-X}$$
$$\frac{125-X}{100} = \frac{132.25}{125-X}$$
$$(125-X)^2 = 13{,}225$$
$$125-X = 115$$
$$X = 125-115 = 10$$
$$i = \frac{25-X}{100} = \frac{25-10}{100} = \frac{15}{100} = 0.15$$

Solution 5.20

Part (i)

The loan outstanding on 9/1/98 by the prospective method is:

$$
\begin{aligned}
B_{9/1/98} &= 1{,}000\left[v^{\frac{10}{12}} + 1.05v^{\frac{14}{12}} + (1.05)^2 v^{\frac{18}{12}} + \cdots + (1.05)^{14} v^{\frac{66}{12}}\right] \\
&= 1{,}000v^{\frac{10}{12}}\left[1 + \left(1.05v^{\frac{4}{12}}\right)^1 + \left(1.05v^{\frac{4}{12}}\right)^2 + \cdots + \left(1.05v^{\frac{4}{12}}\right)^{14}\right] \\
&= 1{,}000v^{\frac{10}{12}}\left[\frac{1-\left[1.05v^{\frac{4}{12}}\right]^{15}}{1-1.05v^{\frac{4}{12}}}\right] \\
&= 1{,}000(0.952603)(18.572034) \\
&= \$17{,}691.77
\end{aligned}
$$

Part (ii)

The loan balance outstanding on 6/30/99 is the loan balance on 9/1/98 accumulated for 10 months since there are no payments during this period:

$$B_{6/30/99} = 17{,}691.77(1.06)^{\frac{10}{12}} = \$18{,}572.04$$

The amount of interest in the first payment is then $I_1 = 18{,}572.04 - 17{,}691.77 = \$880.27$

The amount of principal repaid in the first payment is $P_1 = 1{,}000 - 880.27 = \$119.73$

Part (iii)

The principal outstanding after 6 payments is the present value of future payments occurring after 3/1/01:

$$
\begin{aligned}
B_{3/1/01} &= 1{,}000\left[(1.05)^6 v^{\frac{4}{12}} + (1.05)^7 v^{\frac{8}{12}} + \cdots + (1.05)^{14} v^{\frac{36}{12}}\right] \\
&= 1{,}000(1.05)^6 v^{\frac{4}{12}}\left[1 + \left(1.05v^{\frac{4}{12}}\right)^1 + \cdots + \left(1.05v^{\frac{4}{12}}\right)^8\right] \\
&= 1{,}000(1.05)^6 v^{\frac{4}{12}}\left[\frac{1-\left[1.05v^{\frac{4}{12}}\right]^9}{1-1.05v^{\frac{4}{12}}}\right] \\
&= 1{,}000(1.314318)(10.150941) \\
&= \$13{,}341.57
\end{aligned}
$$

The amount of interest in the seventh payment is then $I_7 = 13{,}341.57\left[(1.06)^{\frac{4}{12}} - 1\right] = \$261.67$

The amount of principal repaid in the seventh payment is $P_7 = 1{,}000(1.05)^6 - 261.67 = \$1{,}078.43$