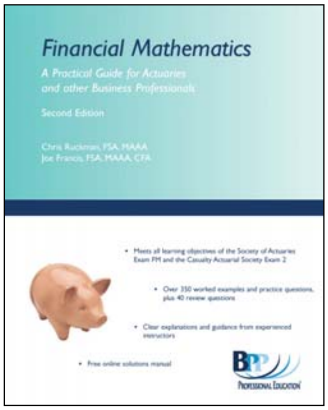

Financial Mathematics
Second Edition

A Practical Guide for Actuaries and other Business Professionals

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Solutions to practice questions – Chapter 8

Solution 8.1

The price of the bonds is found by using each bond's yield to find the present value of the bond's cash flows:

$$P=\sum\frac{CF_t}{(1+y)^t}$$

The price of the 1-year bond is:

$$P=\frac{110}{1.08}=\$101.8519$$

The price of the 2-year bond is:

$$P=\frac{4}{1.08979}+\frac{104}{1.08979^2}=\$91.2389$$

The price of the 3-year bond is:

$$P=\frac{20}{1.09782}+\frac{20}{1.09782^2}+\frac{120}{1.09782^3}=\$125.5085$$

Solution 8.2

The price is the sum of the present values of the cash flows:

$$P=\sum\frac{CF_t}{(1+s_t)^t}=\frac{15}{1.07}+\frac{15}{1.06^2}+\frac{115}{1.08^3}=118.6593$$

The price of the bond is $118.66.

Solution 8.3

We calculated the prices in Solution 8.1, so the yield table with the prices is:

Maturity	Annual coupon	Price	Annual effective yield
1	10.000%	101.8519	8.000%
2	4.000%	91.2389	8.979%
3	20.000%	125.5085	9.782%

We use the following formula to find the spot rates:

$$P = \sum \frac{CF_t}{(1+s_t)^t}$$

Since the 1-year bond has only one cash flow, its yield is equal to the one-year spot rate:

$$s_1 = 8.000\%$$

Plugging s_1 into the formula for the price of the 2-year bond, we solve for s_2:

$$91.2389 = \frac{4}{1.08} + \frac{104}{(1+s_2)^2} \qquad \Rightarrow \qquad s_2 = 9.000\%$$

Plugging s_1 and s_2 into the formula for the price of the 3-year bond, we solve for s_3:

$$125.5085 = \frac{20}{1.08} + \frac{20}{(1.09)^2} + \frac{120}{(1+s_3)^3} \qquad \Rightarrow \qquad s_3 = 10.000\%$$

Solution 8.4

The price per \$100 of par value of the 5-year bond that pays \$7.50 per year is:

$$P = \sum \frac{CF_t}{(1+y)^t} = 7.5a_{\overline{5|}7.664\%} + \frac{100}{1.07664^5} = 99.3394$$

But we cannot use $a_{\overline{5|}7.664\%}$ to value the annuity-immediate consisting of \$40 paid for 5 years, because the yield of 7.664% is only appropriate for the bond with 7.5% coupons. We use the 5-year spot rate to find the value of \$100 in 5 years:

$$\text{Present value of \$100 paid in 5 years} = \frac{100}{1.08^5} = \$68.0583$$

Subtracting the present value of \$100 payable in 5 years from the value of the 7.5% 5-year bond, we are left with the value of a stream of 5 annual payments of \$7.50 each. Once we know the value of 5 annual payments of \$7.50 each, it is a simple matter to scale that value in order to arrive at the value of 5 annual payments of \$40 each:

$$\text{Present value of 5 annual payments of 7.50} = 99.3394 - 68.0583 = 31.2810$$

$$\text{Present value of 5 annual payments of 1.00} = \frac{31.2810}{7.50} = 4.1708$$

$$\text{Present value of 5 annual payments of 40.00} = 40 \times 4.1708 = 166.8322$$

The value of the annuity immediate is \$166.83.

Solution 8.5

The forward rates are used to find the present value of each cash flow:

$$\begin{aligned}P &= \sum \frac{CF_t}{(1+f_0)(1+f_1)\cdots(1+f_{t-1})}\\ &= \frac{15}{1.07} + \frac{15}{(1.07)(1.05009)} + \frac{115}{(1.07)(1.05009)(1.12114)}\\ &= 118.6596\end{aligned}$$

The price of the bond is $118.66.

Solution 8.6

The spot rates from Solution 8.3 are:

Maturity	Annual effective spot rates
1	8.000%
2	9.000%
3	10.000%

The forward rates can be determined based on their relationship with the spot rates:

$$(1+s_1) = (1+f_0) \quad \Rightarrow \quad f_0 = 8.000\%$$

$$f_1 = \frac{(1+s_2)^2}{(1+s_1)^1} - 1 = \frac{(1.09)^2}{(1.08)} = 10.009\%$$

$$f_2 = \frac{(1+s_3)^3}{(1+s_2)^2} - 1 = \frac{(1.10)^3}{(1.09)^2} = 12.028\%$$

Solution 8.7

Each bond has a price of $100 per $100 of par value:

$$P = \sum \frac{CF_t}{(1+f_0)(1+f_1)\cdots(1+f_{t-1})}$$

$$100 = \frac{105}{(1+f_0)} \quad \Rightarrow \quad f_0 = 5.000\%$$

$$100 = \frac{6.5}{1.05} + \frac{106.5}{(1.05)(1+f_1)} \quad \Rightarrow \quad f_1 = 8.122\%$$

$$100 = \frac{8.5}{1.05} + \frac{8.5}{(1.05)(1.08122)} + \frac{108.50}{(1.05)(1.08122)(1+f_2)} \quad \Rightarrow \quad f_2 = 13.212\%$$

Solution 8.8

The following formula is useful for converting forward rates into spot rates:

$$(1+s_t)^t = (1+f_0)(1+f_1)\cdots(1+f_{t-1})$$

Plugging the forward rates into the formula provides us with the spot rates:

$$(1+s_1) = (1.07) \quad \Rightarrow \quad s_1 = 7.000\%$$

$$(1+s_2)^2 = (1.07)(1.05009) \quad \Rightarrow \quad s_2 = 6.000\%$$

$$(1+s_3)^3 = (1.07)(1.05009)(1.12114) \quad \Rightarrow \quad s_3 = 8.000\%$$

Solution 8.9

We normally need to know the coupons in order to find the spot rates from a set of yields, but when the yields do not vary by maturity, the yield curve is flat. A flat yield curve implies that the spot rates are equal to the yields. This is shown below.

We find each spot rate by equating the price calculated from the yields with the price calculated from the spot rates. For simplicity, we assume that the par value of each bond is \$1.00:

Price of bond, calculated with yields = Price of bond, calculated with spots

$$\frac{1+X}{1.10} = \frac{1+X}{(1+s_1)} \quad \Rightarrow \quad s_1 = 10\%$$

$$\frac{Y}{1.10} + \frac{1+Y}{1.10^2} = \frac{Y}{1.10} + \frac{1+Y}{(1+s_2)^2} \quad \Rightarrow \quad s_2 = 10\%$$

$$\frac{Z}{1.10} + \frac{Z}{1.10^2} + \frac{1+Z}{1.10^3} = \frac{Z}{1.10} + \frac{Z}{1.10^2} + \frac{1+Z}{(1+s_3)^3} \quad \Rightarrow \quad s_3 = 10\%$$

Solution 8.10

When the yield curve is flat, the yields, spots, and forwards are equal. This is shown below.

$$(1+s_1) = (1+f_0) \quad \Rightarrow \quad f_0 = 10.000\%$$

$$f_1 = \frac{(1+s_2)^2}{(1+s_1)^1} - 1 = \frac{(1.10)^2}{(1.10)} = 10.000\%$$

$$f_2 = \frac{(1+s_3)^3}{(1+s_2)^2} - 1 = \frac{(1.10)^3}{(1.00)^2} = 10.000\%$$

Solution 8.11

The spot rates can be calculated through iterative use of the formula for the price of a bond:

$$P = \sum \frac{CF_t}{(1+s_t)^t}$$

$$106.7961 = \frac{110}{1+s_1} \quad \Rightarrow \quad s_1 = 3.000\%$$

$$94.4588 = \frac{2}{1.03} + \frac{102}{(1+s_2)^2} \quad \Rightarrow \quad s_2 = 5.000\%$$

$$100.7571 = \frac{8}{1.03} + \frac{8}{(1.05)^2} + \frac{108}{(1+s_3)^3} \quad \Rightarrow \quad s_3 = 8.000\%$$

The spot rates can now be used to find the price of the 3-year 12% bond:

$$P = \sum \frac{CF_t}{(1+s_t)^t} = \frac{12}{1.03} + \frac{12}{1.05^2} + \frac{112}{1.08^3} = 111.4440$$

The price of the bond is $111.44.

Solution 8.12

The forward rate can be found with the following formula:

$$f_3 = \frac{(1+s_4)^4}{(1+s_3)^3} - 1$$

The 3-year and 4-year spot rates are derived from the prices of the 3-year and 4-year zero-coupon bonds:

$$81.6298 = \frac{100}{(1+s_3)^3} \quad \Rightarrow \quad s_3 = 7.000\%$$

$$73.5030 = \frac{100}{(1+s_4)^4} \quad \Rightarrow \quad s_4 = 8.000\%$$

The values for the spot rates are then plugged into the formula for the forward rate:

$$f_3 = \frac{(1+s_4)^4}{(1+s_3)^3} - 1 = \frac{1.08^4}{1.07^3} - 1 = 0.110564$$

The forward rate applicable to the fourth year is 11.0564%.

Solution 8.13

We are given $s_1 = 5\%$. Using $s_1 = f_0$, we find that:

$$(1+s_2)^2 = (1+f_0)(1+f_1)$$

$$(1+s_2)^2 = (1.05)(1.07)$$

The 3-year bond is priced at par. The formula for the price of the bond in terms of the spot rates is used to determine s_3:

$$100 = \frac{8}{(1+s_1)} + \frac{8}{(1+s_2)^2} + \frac{108}{(1+s_3)^3}$$

$$100 = \frac{8}{1.05} + \frac{8}{(1.05)(1.07)} + \frac{108}{(1+s_3)^3}$$

$$s_3 = 8.1996\%$$

The 3-year spot rate is 8.1996%.

Solution 8.14

The yield of the bond can be used to determine its price:

$$P = \frac{5.50}{1.093} + \frac{105.50}{1.093^2} = 93.3425$$

So the price of the bond is \$93.3425.

The price that is consistent with the spot rates is:

$$P = \frac{5.50}{1.07} + \frac{105.50}{1.09^2} = 93.9374$$

Since the price of \$93.3425 is lower than the price of \$93.9374 indicated by the spot rates, the bond is undervalued, indicating that we can obtain arbitrage profits by purchasing the bond.

Buying the bond provides cash flows of \$5.50 at time 1 and \$105.50 at time 2. In order to obtain arbitrage profits we must find a way to sell those cash flows for more than the \$93.3425 that they cost. This can be accomplished by borrowing the present value of \$5.50 payable in time 1 and \$105.50 payable in time 2:

$$\frac{5.50}{1.07} = 5.1402 \quad \text{and} \quad \frac{105.50}{1.09^2} = 88.7972$$

So \$5.1402 is borrowed at the 1-year spot rate and \$88.7972 is borrowed at the 2-year spot rate. Borrowing these funds provides \$93.9374 now:

$$5.1402 + 88.7972 = 93.9374$$

Therefore, at time 0, we have \$93.9374 from the borrowing and we must spend \$93.3425 to purchase the bond, leaving us with \$0.5949:

$$93.9374 - 93.3425 = 0.5949$$

The table below illustrates the cash flows at time 0, 1, and 2.

End of Year	Cash flow from long position	Cash flow from short position	Net cash flow
0	−93.3425	93.9374	0.5949
1	5.5000	−5.5000	0.0000
2	105.5000	−105.5000	0.0000

Since there are no net cash flows occurring at time 1 and time 2, the only net cash flow is the net receipt of \$0.5949 at time 0.

Solution 8.15

Part (i)

The bond prices are:

$$P = \frac{112}{1.12} = 100 \quad \text{and} \quad P = \frac{10}{1.14847} + \frac{110}{1.14847^2} = 92.1048$$

The price of the 1-year bond is \$100.0000 and the price of the 2-year bond is \$92.1048.

Since the 1-year bond has just one cash flow, the 1-year spot rate is equal to the bond's yield:

$$100 = \frac{112}{1+s_1} \quad \Rightarrow \quad s_1 = 12\%$$

Once the 1-year spot rate is determined, the 2-year bond is used to determine the 2-year spot rate:

$$92.1048 = \frac{10}{1.12} + \frac{110}{(1+s_2)^2} \qquad \Rightarrow \qquad s_2 = 15.000\%$$

The 2-year spot rate is 15.000%.

Part (ii)

The investor wants to deposit $1,000 now and have it earn 15% for 2 years:

$$1,000(1.15)^2 = \$1,322.50$$

Since the investor wants a positive cash flow of $1,322.50 at time 2, the investor should purchase enough of the 2-year bond to produce this cash flow at time 2:

$$\frac{1,322.50}{110} = 12.0227 \qquad \Rightarrow \qquad \text{purchase 12.0227 2-year bonds}$$

Purchasing the 2-year bonds provides the proper cash flow at time 2, but it also provides cash flow at time 1.

$$\text{Coupons at time 1 from 2-year bond} = 12.0227(10) = 120.2273$$

Selling 1-year bonds short eliminates this cash flow:

$$\frac{120.2273}{112} = 1.0735 \qquad \Rightarrow \qquad \text{Sell 1.0735 1-year bonds}$$

The investor's objective is accomplished by purchasing 12.0227 2-year bonds and selling 1.0735 1-year bonds.

At time 0, the investor's net cash flow is:

$$-12.0227 \times 92.1048 + 1.0735 \times 100 = -\$1,000.00$$

At time 1, the investor's net cash flow is:

$$12.0227 \times 10 - 1.0735 \times 112 = \$0.00$$

At time 2, the investor's net cash flow is:

$$12.0227 \times 110 = \$1,322.50$$

The table below illustrates the cash flows at time 0, 1, and 2.

End of Year	Cash flow from long position	Cash flow from short position	Net cash flow
0	−1,107.35	107.35	−1,000.00
1	120.23	−120.23	0.00
2	1,322.50	0.00	1,322.50

The net cash flows in the far right column are the cash flows obtained by depositing $1,000 for two years at 15% interest.

Solution 8.16

In Solution 8.15, the investor earned 15% on a deposit. By simply taking the opposite of the actions for earning 15% on a deposit, the investor can borrow at 15%. Since the institution is offering 17%, the investor can borrow at 15% and invest at 17%.

The institution offering 17% on a 2-year deposit can be viewed as providing a lump sum at time 2 for a cost of $1,000 now:

$$\text{Lump sum at time 2} = 1,000 \times 1.17^2 = \$1,368.90$$

To be consistent with the spot rate of 15% calculated previously, the value of the lump sum would have to be:

$$\frac{1,368.90}{1.15^2} = \$1,035.0851$$

Since this asset is priced at $1,000 by the institution, it is undervalued, and the investor buys the asset by depositing $1,000 with the institution.

In Solution 8.15, we saw how to invest $1,000 now in order to receive $1,322.50 in two years. The amount that must be invested now in order to receive $1,368.90 in two years is therefore:

$$\frac{1,368.90}{1,322.50} \times 1,000 = \$1,035.0851$$

By scaling and reversing the transactions of Solution 8.15, the investor borrows $1,035.0851 now in exchange for a promise to pay $1,368.90 in two years.

$$12.0227 \times \frac{1,035.0851}{1,000.000} = 12.4445 \quad \Rightarrow \quad \text{sell 12.4445 2-year bonds}$$

$$1.0735 \times \frac{1,035.0851}{1,000.000} = 1.1111 \quad \Rightarrow \quad \text{purchase 1.1111 1-year bonds}$$

The sale of 12.4445 2-year bonds results in the following cash flows:

$$\text{Time 0:} \quad 12.4445 \times 92.1048 = 1,146.20$$
$$\text{Time 2:} \quad -12.4445 \times 10 = 124.44$$
$$\text{Time 3:} \quad -12.4445 \times 110 = 1,368.90$$

The purchase of 1.1111 1-year bonds results in the following cash flows:

$$\text{Time 0:} \quad -1.1111 \times 100.00 = -111.11$$
$$\text{Time 2:} \quad 1.1111 \times 112 = 124.45$$

The table below illustrates the cash flows at time 0, 1, and 2.

End of Year	Cash flow from investing $1,000 with institution	Cash flow from sale of 2-year bond	Cash flow from purchase of 1-year bond	Net cash flow
0	−1,000.00	1,146.19	−111.11	35.08
1	0.00	−124.45	124.45	0.00
2	1,368.90	−1,368.90	0.00	0.00

The far right column illustrates that the only net cash flow occurs at time 0 and that net cash flow is the net receipt of $35.08.

Solution 8.17

We have:

$$\frac{s_1^{(2)}}{2} = 0.050 \qquad \frac{s_2^{(2)}}{2} = 0.075 \qquad \frac{s_3^{(2)}}{2} = 0.100$$

The price is:

$$P = \sum \frac{CF_t}{\left(1+\frac{s_t^{(m)}}{m}\right)^{mt}} = \frac{4}{1.05^2} + \frac{4}{1.075^4} + \frac{104}{1.10^6} = 65.3286$$

The present value of the 3-year bond is $65.33.

Solution 8.18

The following formula can be used to convert forward rates into spot rats:

$$\left(1+\frac{s_t^{(m)}}{m}\right)^{mt} = \left(1+\frac{f_0^{(p)}}{p}\right)^p \left(1+\frac{f_1^{(p)}}{p}\right)^p \cdots \left(1+\frac{f_{t-1}^{(p)}}{p}\right)^p$$

Applying the formula:

$$\left(1+\frac{s_1^{(4)}}{4}\right)^4 = \left(1+\frac{0.06}{2}\right)^2 \qquad \Rightarrow \qquad s_1^{(4)} = 5.9557\%$$

$$\left(1+\frac{s_2^{(4)}}{4}\right)^8 = \left(1+\frac{0.06}{2}\right)^2 \left(1+\frac{0.05}{2}\right)^2 \qquad \Rightarrow \qquad s_2^{(4)} = 5.4621\%$$

$$\left(1+\frac{s_3^{(4)}}{4}\right)^{12} = \left(1+\frac{0.06}{2}\right)^2 \left(1+\frac{0.05}{2}\right)^2 \left(1+\frac{0.07}{2}\right)^2 \qquad \Rightarrow \qquad s_3^{(4)} = 5.9541\%$$

Solution 8.19

Forward rates, like all interest rates, exhibit the following relationship between the effective rate for a period of length $\frac{1}{p}$ and the annual effective rate:

$$\left(1+\frac{i^{(p)}}{p}\right)^p = 1+i$$

Therefore, the monthly effective forward rates have the following relationship with the annual effective forward rates:

$$\left(1+\frac{f^{(12)}_{t,t+\frac{1}{12}}}{12}\right)^{12} = \left(1+f_{t,t+\frac{1}{12}}\right)$$

$$\left(1+\frac{f^{(12)}_{t,t+\frac{1}{12}}}{12}\right) = \left(1+f_{t,t+\frac{1}{12}}\right)^{\frac{1}{12}}$$

In the formula below, we set $h=\frac{1}{12}$: and solve for $s_{0.5}$:

$$(1+s_t)^t = \left(1+f_{0,h}\right)^h \left(1+f_{h,2h}\right)^h \cdots \left(1+f_{t-h,t}\right)^h$$

$$(1+s_t)^t = \left(1+f_{0,\frac{1}{12}}\right)^{\frac{1}{12}} \left(1+f_{\frac{1}{12},\frac{2}{12}}\right)^{\frac{1}{12}} \cdots \left(1+f_{t-\frac{1}{12},t}\right)^{\frac{1}{12}}$$

$$(1+s_{0.5})^{0.5} = \left(1+\frac{f^{(12)}_{0,\frac{1}{12}}}{12}\right)\left(1+\frac{f^{(12)}_{\frac{1}{12},\frac{2}{12}}}{12}\right)\cdots\left(1+\frac{f^{(12)}_{\frac{5}{12},\frac{6}{12}}}{12}\right)$$

$$(1+s_{0.5})^{0.5} = (1.01)(1.015)(1.008)(1.02)(1.005)(1.013)$$

$$s_{0.5} = 15.146\%$$

The same logic gives us the following formula for s_1:

$$(1+s_1)^1 = \left(1+\frac{f^{(12)}_{0,\frac{1}{12}}}{12}\right)\left(1+\frac{f^{(12)}_{\frac{1}{12},\frac{2}{12}}}{12}\right)\cdots\left(1+\frac{f^{(12)}_{\frac{11}{12},\frac{12}{12}}}{12}\right)$$

$$(1+s_1)^1 = (1.01)(1.015)(1.008)(1.02)(1.005)(1.013)(1.01)(1.011)(1.012)(1.006)(1.002)(1.009)$$

$$s_1 = 12.780\%$$

Finally, we convert the annual effective spot rates into semiannually compounded spot rates:

$$s^{(2)}_{0.5} = 2\left[(1+s_{0.5})^{0.5}-1\right] = 14.612\%$$

$$s^{(2)}_{1} = 2\left[(1+s_{1})^{0.5}-1\right] = 12.396\%$$

The 0.5-year semiannually compounded spot rate is 14.612%, and the 1-year semiannually compounded spot rate is 12.39%.

Solution 8.20

We are asked to find $\frac{f_{1,2}^{(2)}}{2}$. If we set $h = 1$, then the following formula determines the annual effective forward rate in terms of the annual effective spot rates:

$$f_{t-h,t} = \left(\frac{(1+s_t)^t}{(1+s_{t-1})^{t-h}} \right)^{\frac{1}{h}} - 1$$

$$f_{1,2} = \left(\frac{(1+s_2)^2}{(1+s_1)} \right) - 1$$

The values of s_1 and s_2 are related to the semiannual spot rates as follows:

$$(1+s_1) = \left(1 + \frac{s_{1.0}^{(2)}}{2}\right)^2 = \left(1 + \frac{0.135}{2}\right)^2 \qquad \text{and} \qquad (1+s_2) = \left(1 + \frac{s_{2.0}^{(2)}}{2}\right)^2 = \left(1 + \frac{0.16}{2}\right)^2$$

Therefore:

$$f_{1,2} = \left(\frac{\left(1 + \frac{0.16}{2}\right)^{2\times 2}}{\left(1 + \frac{0.135}{2}\right)^2} \right) - 1 = \frac{1.08^4}{1.0675^2} - 1 = 19.3876\%$$

Converting the forward rate into a rate that is effective over six months, we have:

$$\frac{f_{1,2}^{(2)}}{2} = 1.193876^{0.5} - 1 = 9.2646\%$$

The 1-year implied forward rate for the second year, expressed as a rate that is effective over six months is 9.2646%.