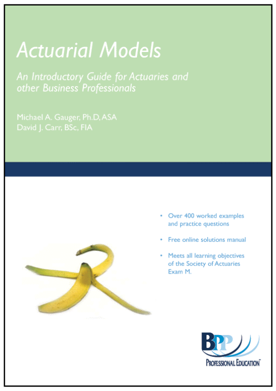

PROFESSIONAL EDUCATION™

Actuarial models

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Solutions to practice questions – Chapter 1

Solution 1.1

$$ {}_2p_0 = \frac{l_2}{l_0} = \frac{985}{1,000}\ ,\ {}_{2|}q_0 = \frac{d_2}{l_0} = \frac{3}{1,000}\ ,\ {}_{4|\,2}q_3 = \frac{l_7 - l_9}{l_3} = \frac{9}{982} $$

$$ p_4 = \frac{l_5}{l_4} = \frac{976}{979}\ ,\ q_5 = \frac{d_5}{l_5} = \frac{4}{976} $$

Solution 1.2

The probability ${}_{2|}q_1 = 0.015$ is the probability that a life currently age 1 will die between ages 3 and 4. View each of the 20 lives age 1 in the group as a trial, where 'success' means that the life dies between ages 3 and 4. Then the random number of deaths follows a binomial distribution with $n = 20$ trials and $p = {}_{2|}q_1 = 0.015$. The expected number of deaths is $np = 0.30$ and the variance in the number of deaths is $npq = 0.29550$.

Solution 1.3

The probability function is: $\Pr(K = k) = d_k / l_0$. From the table we have:

$l_0 = 100$, $d_0 = l_0 - l_1 = 54$, $d_1 = 27$, $d_2 = 13$, $d_3 = 6$

So it follows that $\Pr(K = 0) = 54/100$, $\Pr(K = 1) = 27/100$, and so on.

Solution 1.4

If you are given a formula for the force function, then the survival function is the obvious choice for the first calculation:

$$ \mu(x) = \frac{3}{2+x} \Rightarrow \int_0^x \mu(y)\,dy = \left(3\ln(2+y)\right)\Big|_0^x = \left(\ln\left(\frac{2+x}{2}\right)\right)^3 \Rightarrow $$

$$ s_X(x) = \exp\left(-\int_0^x \mu(y)\,dy\right) = \left(\frac{2}{2+x}\right)^3 \Rightarrow F_X(x) = 1 - s_X(x) = 1 - \left(\frac{2}{2+x}\right)^3 \Rightarrow $$

$$ f_X(x) = F_X'(x) = \frac{2^3 \times 3}{(2+x)^4} $$

Solution 1.5

$$p_1 = \frac{l_2}{l_1} = \frac{s_X(2)}{s_X(1)} = \frac{\left(2/(2+2)\right)^3}{\left(2/(2+1)\right)^3} = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

$$_{2|}q_1 = \frac{l_3 - l_4}{l_1} = \frac{s_X(3) - s_X(4)}{s_X(1)} = \frac{\left(\frac{2}{5}\right)^3 - \left(\frac{2}{6}\right)^3}{\left(\frac{2}{3}\right)^3} = \frac{91}{1,000}$$

Solution 1.6

$$\mu(x) = \frac{0.5}{100-x} \Rightarrow \int_0^x \mu(y)\,dy = 0.5\left(-\ln(100-y)\Big|_0^x\right) = \ln\left(\left(\frac{100}{100-x}\right)^{0.5}\right) \Rightarrow$$

$$s_X(x) = \exp\left(-\ln\left(\left(\frac{100}{100-x}\right)^{0.5}\right)\right) = \left(\frac{100-x}{100}\right)^{0.5} \Rightarrow F_X(x) = 1 - \left(\frac{100-x}{100}\right)^{0.5} \Rightarrow$$

$$f_X(x) = \frac{1}{20(100-x)^{0.5}}$$

Solution 1.7

$$s_X(x) = \left(\frac{100-x}{100}\right)^{0.5} \Rightarrow {}_{20}p_{40} = \frac{s_X(60)}{s_X(40)} = \sqrt{\frac{40}{60}} = 0.81650$$

$$_{20|20}q_{40} = {}_{20}p_{40} \times (1 - {}_{20}p_{60}) = \sqrt{\frac{40}{60}} \times \left(1 - \sqrt{\frac{20}{40}}\right) = 0.23915$$

Solution 1.8

$$\mu(x) = \frac{1.1}{100+x} \Rightarrow \int_0^x \mu(y)\,dy = 1.1\ln\left(\frac{100+x}{100}\right) \Rightarrow$$

$$s_X(x) = \exp\left(-1.1\ln\left(\frac{100+x}{100}\right)\right) = \left(\frac{100}{100+x}\right)^{1.1} \Rightarrow$$

$$_tp_{20} = \frac{s_X(20+t)}{s_X(20)} = \left(\frac{100}{100+20+t}\right)^{1.1} \Big/ \left(\frac{100}{100+20}\right)^{1.1} = \left(\frac{120}{120+t}\right)^{1.1}$$

Solution 1.9

For the curtate lifetime at age 20 to be less than 2, we must have $K(20) = 0$ or 1. That means that (20) must die within 2 years. The probability is 1 minus the probability of surviving the next 2 years:

$$_2q_{20} = \Pr\left(K(20) = 0 \text{ or } 1\right) = 1 - s_{T(20)}(2) = 1 - \left(\frac{120}{122}\right)^{1.1} = 0.01802$$

Solution 1.10

$$\mu(x) = 0.015 \quad \Rightarrow \quad s_X(x) = \exp\left(-\int_0^x 0.015\,dy\right) = e^{-0.015x} \quad \Rightarrow$$

$$f_{T(20)}(t) = {}_tp_{20}\;\mu(20+t) = \frac{s_X(20+t)}{s_X(20)} \times \mu(20+t) = \frac{0.015e^{-0.015(20+t)}}{e^{-0.015\times 20}} = 0.015e^{-0.015t}$$

Solution 1.11

Start by computing a formula for the probability function:

$$\Pr\left(K(20) = k\right) = {}_{k|}q_{20} = {}_kp_{20} - {}_{k+1}p_{20} = e^{-0.015(k)} - e^{-0.015(k+1)} = \left(1 - e^{-0.015}\right)\left(e^{-0.015k}\right)$$

The probability that the curtate lifetime exceeds 1 is:

$$1 - \Pr\left(K(20) = 0\right) - \Pr\left(K(20) = 1\right) = 1 - \left(1 - e^{-0.015}\right) - \left(1 - e^{-0.015}\right)e^{-0.015} = 0.97045$$

Note: You could have just as easily used $\Pr\left(K(20) \ge 2\right) = e^{-2\times 0.015}$.

Solution 1.12

$$l_x = 1{,}000(100-x)^{0.95} \quad \Rightarrow \quad T_x = \int_x^{100} l_y\,dy = 1{,}000\left(-\frac{(100-x)^{1.95}}{1.95}\right)\Bigg|_x^{100} = \frac{1{,}000(100-x)^{1.95}}{1.95}$$

Solution 1.13

Use the formula developed in Solution 1.12 and the fact that $T_{20} - T_{25}$ is the number of people-years between ages 20 and 25 lived by the l_{20} lives that survive to age 20:

$$\mathring{e}_{20:\overline{5}|} = \frac{T_{20} - T_{25}}{l_{20}} = \frac{\dfrac{1{,}000(80)^{1.95}}{1.95} - \dfrac{1{,}000(75)^{1.95}}{1.95}}{1{,}000(80)^{0.95}} = \frac{80^{1.95} - 75^{1.95}}{1.95 \times 80^{0.95}} = 4.85141$$

Solution 1.14

$$\mu(x)=0.015 \;\Rightarrow\; {}_tp_x=e^{-0.015t} \;\Rightarrow\; \mathring{e}_{x:\overline{1}|}=\int_0^1 {}_tp_x\,dt=\frac{1-e^{-0.015}}{0.015}=0.99254 \;\Rightarrow$$

$$a(x)=\frac{\mathring{e}_{x:\overline{1}|}-p_x}{q_x}=\frac{0.99254-e^{-0.015}}{1-e^{-0.015}}=0.49875$$

Note: When the force of mortality is constant, on average, deaths during a given year of age occur slightly before mid-year.

Solution 1.15

$$q_{20}=1-p_{20}=1-e^{-0.015}=0.01489$$

$$m_{20}=\frac{\int_0^1 {}_tp_{20}\,\mu(20+t)\,dt}{\int_0^1 {}_tp_{20}\;dt}=\frac{\int_0^1 {}_tp_{20}\,0.015\,dt}{\int_0^1 {}_tp_{20}\;dt}=0.01500$$

Note. When the force of mortality is constant, the central rate at age x is the same as the force.

Solution 1.16

$$l_x=1{,}000(100-x)^{0.75} \;\Rightarrow\; \mu(x)=\frac{-l'_x}{l_x} \;\Rightarrow\; l_x\mu(x)=-l'_x=750/(100-x)^{0.25}$$

Solution 1.17

$$q_{30}=0.01\,,\; q_{31}=0.02\,,\; q_{32}=0.03 \;\text{ and the UDD }\;\Rightarrow$$

$$\mu(31.4)=\mu(31+0.4)=\frac{q_{31}}{1-0.4q_{31}}=\frac{0.02}{1-0.4\times0.02}=0.02016$$

$${}_{1.4}p_{30}=p_{30}\times{}_{0.4}p_{31}=(1-0.01)(1-0.4\times0.02)=0.98208$$

$$f_{T(30)}(1.4)={}_{1.4}p_{30}\,\mu(31.4)=0.01980$$

Solution 1.18

A person age 30 who survives 3 years gets credit for 3 years of life in both $e_{30:\overline{3}|}$ and $\mathring{e}_{30:\overline{3}|}$. A person who dies during the next 3 years gets more credit in the calculation of $\mathring{e}_{30:\overline{3}|}$ for part of the final year of life. Since $a(x)=0.5$ for all ages under the UDD assumption, it seems reasonable that $\mathring{e}_{30:\overline{3}|}=e_{30:\overline{3}|}+0.5\,{}_3q_{30}$.

$$e_{30:\overline{3}|}=p_{30}+{}_2p_{30}+{}_3p_{30}=0.99+(0.99\times0.98)+(0.99\times0.98\times0.97)=2.90129$$

$$\mathring{e}_{30:\overline{3}|}=e_{30:\overline{3}|}+0.5\,{}_3q_{30}=2.90129+0.5\times(1-0.99\times0.98\times0.97)=2.93075$$

Solution 1.19

$$l_x = 1{,}000e^{-0.02x} \;\Rightarrow\; p_x = l_{x+1}/l_x = e^{-0.02} \text{ for all } x$$

$$\begin{aligned}
{}_3q_{[20]} &= 1 - {}_3p_{[20]} = 1 - p_{[20]} \times p_{[20]+1} \times p_{22} \\
&= 1 - (1 - 0.8q_{20})(1 - 0.9q_{21})p_{22} \\
&= 1 - \left(1 - 0.8(1 - p_{20})\right)\left(1 - 0.9(1 - p_{21})\right)p_{22} \\
&= 1 - \left(1 - 0.8\left(1 - e^{-0.02}\right)\right)\left(1 - 0.9\left(1 - e^{-0.02}\right)\right)p_{22} \\
&= 0.05252
\end{aligned}$$

Solution 1.20

$$l_{[20]} = \frac{l_{22}}{{}_2p_{[20]}} = \frac{1{,}000\,e^{-0.02\times 22}}{\left(1 - 0.8\left(1 - e^{-0.02}\right)\right)\left(1 - 0.9\left(1 - e^{-0.02}\right)\right)} = 666.28$$