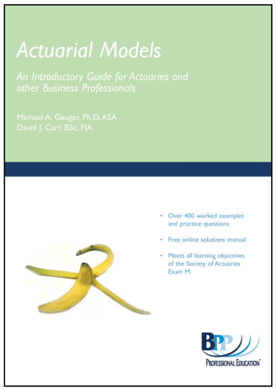

PROFESSIONAL EDUCATION

Actuarial models

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Solutions to practice questions – Chapter 12

Solution 12.1

According to Theorem 12.1, the new severity model is a weighted average of the two component severity models:

$$f_X(x) = \frac{15}{20} f_1(x) + \frac{5}{20} f_2(x)$$

where:

$$f_1(x) = 0.001 \quad \text{for} \quad 0 < x \le 1{,}000 \quad \text{(zero otherwise)}$$

$$f_2(x) = \frac{3(2{,}000)^3}{(2{,}000 + x)^4} \quad \text{for } x > 0 \qquad \text{(zero otherwise)}$$

Solution 12.2

The expected loss is:

$$\begin{aligned} E[X] &= \frac{15}{20} E[X_1] + \frac{5}{20} E[X_2] \quad \text{where } X_1 \sim U(0,1{,}000),\ X_2 \sim \text{Pareto } \alpha = 3,\ \theta = 2{,}000 \\ &= \frac{15}{20} \times \frac{1{,}000}{2} + \frac{5}{20} \times \frac{2{,}000}{3-1} = 625 \end{aligned}$$

So the probability that a loss exceeds the expected value is:

$$\begin{aligned} \Pr(X > 625) &= \int_{625}^{\infty} f_X(x)\,dx = \int_{625}^{\infty} \left(0.75 f_{X_1}(x) + 0.25 f_{X_2}(x)\right) dx \\ &= 0.75 \int_{625}^{\infty} f_{X_1}(x)\,dx + 0.25 \int_{625}^{\infty} f_{X_2}(x)\,dx \\ &= 0.75\, s_{X_1}(625) + 0.25\, s_{X_2}(625) \\ &= 0.75\left(1 - \frac{625}{1{,}000}\right) + 0.25\left(\frac{2{,}000}{2{,}000+625}\right)^3 = 0.39182 \end{aligned}$$

Solution 12.3

The expected aggregate annual loss is:

$$E[S] = E[N_1]\,E[X_1] + E[N_2]\,E[X_2] = 15 \times \frac{1{,}000}{2} + 5 \times \frac{2{,}000}{3-1} = 12{,}500$$

It could also have been computed using the result of Question 12.2: $E[S] = E[N]\,E[X] = 20 \times 625 = 12{,}500$. The compound Poisson variance formula from Theorem 12.1 results in:

$$\operatorname{var}(S) = \lambda_1\, E\left[X_1^2\right] + \lambda_2\, E\left[X_2^2\right] = 15 \times \frac{1{,}000^2}{3} + 5 \times \frac{2{,}000^2 \times 2!}{(3-1)(2-1)} = 25 \text{ million}$$

Approximating the distribution of S by a normal distribution with $\mu = 12{,}500$, $\sigma = 5{,}000$, we have:

$$\begin{aligned}\Pr\big(S \geq 1.5E[S]\big) &= \Pr(S \geq 18{,}750) \approx 1 - \Phi\left(\frac{18{,}750 - 12{,}500}{5{,}000}\right) = 1 - \Phi(1.25) \\ &\approx 1 - \frac{0.8849 + 0.9032}{2} = 0.106\end{aligned}$$

Solution 12.4

For the lognormal approximation, we assume that $S \approx L = e^{N\left(\mu,\sigma^2\right)}$ where:

$$\begin{aligned}12{,}500 &= E[S] = E[L] = e^{\mu + 0.5\sigma^2} \\ 25{,}000{,}000 + 12{,}500^2 &= E\left[S^2\right] = E\left[L^2\right] = e^{2\mu + 2\sigma^2}\end{aligned}$$

Square the first equation, take natural log of both sides of both equations, and then subtract the first equation from the second. The result is:

$$\sigma^2 = 0.14842 \quad , \quad \mu = 9.35927$$

Approximating the distribution of S by a lognormal distribution with $\sigma^2 = 0.14842$, $\mu = 9.35927$, we have:

$$\begin{aligned}\Pr\big(S \geq 1.5E[S]\big) &= \Pr(S \geq 18{,}750) \approx \Pr\Big(N\left(\mu,\sigma^2\right) \geq \ln(18{,}750) = 9.83895\Big) \\ &= 1 - \Phi\left(\frac{9.83895 - \mu}{\sigma}\right) = 1 - \Phi(1.24509)\end{aligned}$$

So the answer is virtually the same as in Solution 12.3.

Solution 12.5

From Theorem 10.1, we have:

$$250 = E[Y] = E[Z]\underbrace{\Pr(Y>0)}_{0.80} \;\Rightarrow\; E[Z] = 312.50 \text{ (the expected payment per claim)}$$

From the relation $S = Y_1 + \cdots + Y_{N_L} = Z_1 + \cdots + Z_{N_P}$, we also have:

$$\underbrace{E[N_L]}_{10}\,\underbrace{E[Y]}_{250} = E[S] = E[N_P]\underbrace{E[Z]}_{312.50} \;\Rightarrow\; E[N_P] = 8 \text{ (the expected annual number of claims)}$$

Solution 12.6

Aggregate annual claims follow a compound Poisson distribution:

$$S = (X_1 - 100)_+ + \cdots + \left(X_{N_L} - 100\right)_+ \quad \text{where } N_L \sim \text{Poisson } \lambda = 10$$

The severity model X follows an exponential distribution with mean $\theta = 500$. From Tables 10.3 and 10.4, we have:

$$E[X \wedge d] = \theta\left(1 - e^{-d/\theta}\right) = 500\left(1 - e^{-100/500}\right) = 90.63462 \qquad \text{(Table 10.3)}$$

$$E\left[(X \wedge d)^2\right] = 2\theta^2\Gamma(3;\, d/\theta) + d^2 e^{-d/\theta} \qquad \text{(Table 10.4)}$$

$$= 2(500)^2\,\Gamma(3;\, 0.2) + 100^2 e^{-0.2}$$

$$= 500{,}000\left(1 - e^{-0.2}\left(1 + 0.2 + \frac{0.2^2}{2!}\right)\right) + 8{,}187.30753 = 8{,}761.54813$$

From Theorem 10.2 and Theorem 10.4(ii), we have:

$$E\left[(X-100)_+\right] = E[X] - E[X \wedge 100] = 500 - 90.63462 = 409.36538$$

$$E\left[(X-100)_+^2\right] = E\left[X^2\right] - E\left[(X \wedge 100)^2\right] - 2\times 100\,E\left[(X-100)_+\right]$$

$$= 2(500)^2 - 8{,}761.54813 - 200 \times 409.36538 = 409{,}365.3766$$

These moments could have been calculated more quickly by realizing that $Z = X - 100 \mid X > 100$ is also exponentially distributed with parameter $\theta = 500$ (see Table 10.2):

$$E\left[(X-100)_+\right] = E[Z]\Pr(Y > 0) = 500 \times e^{-100/500} = 409.36538$$

$$E\left[(X-100)_+^2\right] = E\left[Z^2\right]\Pr(Y > 0) = 2 \times 500^2 \times e^{-100/500} = 409{,}365.3766$$

So from Theorem 12.1, we have:

$$E[S] = \lambda E\left[(X-100)_+\right] = 4{,}093.65 \quad , \quad \text{var}(S) = \lambda E\left[(X-100)_+^2\right] = 4{,}093{,}653.77$$

Solution 12.7

We assume that S is approximately normal in distribution with mean $\mu = 4,093.65$ and variance $\sigma^2 = 4,093,653.77$. Now we need the expected limited loss formula for a normal distribution:

$$E[S \wedge d] = (\mu - d)\ \Phi\left(\frac{d-\mu}{\sigma}\right) + d - \frac{\sigma}{\sqrt{2\pi}} \exp\left[-½\left(\frac{d-\mu}{\sigma}\right)^2\right]$$

$$= (4,093.65 - 7,500)\Phi\left(\frac{7,500 - 4,093.65}{\sqrt{4,093,653.77}}\right) + 7,500 - \sqrt{\frac{4,093,653.77}{2\pi}} \times \exp\left(-\frac{(7,500 - 4,093.65)^2}{2 \times 4,093,653.77}\right)$$

$$= -3,406.35\ \underbrace{\Phi(1.68)}_{\approx 0.9534} + 7,500 - 807.17 \times 0.24239 = 4,056.74$$

$$E\left[(S - 7,500)_+\right] = E[S] - E[S \wedge 7,500] = 36.91$$

Solution 12.8

The lognormal approximation takes a little more effort:

$$4{,}093.65 = E[S] = e^{\mu + 0.5\sigma^2} \quad , \quad 20,851,655 = E\left[S^2\right] = e^{2\mu + 2\sigma^2}$$

$$\Rightarrow \quad \sigma^2 = 0.21856 \quad , \quad \mu = 8.20791$$

Now we need the expected limited loss formula for the lognormal distribution:

$$E\left[S \wedge d\right] = e^{\mu + 0.5\sigma^2}\ \Phi\left(\frac{\log(d) - \mu - \sigma^2}{\sigma}\right) + d\left(1 - \Phi\left(\frac{\log(d) - \mu}{\sigma}\right)\right)$$

$$= 4,093.65\ \Phi\left(\frac{8.92266 - 8.20791 - 0.21856}{\sqrt{0.21856}}\right) + 7,500\left(1 - \Phi\left(\frac{8.92266 - 8.20791}{\sqrt{0.21856}}\right)\right)$$

$$= 4,093.65\ \underbrace{\Phi(1.061)}_{0.8553} + 7,500\left(1 - \underbrace{\Phi(1.529)}_{0.9369}\right) = 3,976.32$$

$$E\left[(S - 7,500)_+\right] = E[S] - E[S \wedge 7,500] = 117.32$$

Solution 12.9

From Theorem 11.2 and the table following this theorem, we know that N_P follows a Poisson distribution with parameter λ^* that is closely related to the parameter λ:

$$\upsilon = \Pr(X > d) = s_X(d) = s_X(100) = e^{-100/500} = 0.81873$$

$$\lambda^* = \lambda\upsilon = 10 \times 0.81873 = 8.18731$$

Solution 12.10

If n is the number of policies in Year 2004, and n_1 is the number of policies in Year 2005, we are given that $1.10 = n_1 / n$, a 10% increase in exposure. From results in Section 5 of Chapter 11, it follows that the model for the frequency of losses in Year 2005, $N_{05:L}$, is Poisson with parameter:

$$\lambda^* = n_1 \lambda / n = 1.10 \times 10 = 11$$

Solution 12.11

The loss model in Year 2005 is $X_{05} = 1.03X$. Since the parameter θ in an exponential distribution is a scale parameter, it follows that X_{05} follows an exponential distribution with mean $\theta^* = 1.03\theta = 515$. We have seen in Solution 12.10 that the frequency of losses in Year 2005, $N_{05:L}$ is Poisson with parameter $\lambda^* = 11$.
From Theorem 11.2 and the following table in Section 6 of Chapter 11, we know that the frequency of payment events in Year 2005 is Poisson with parameter:

$$\lambda^{**} = \upsilon\lambda^* = \Pr\left(X_{05} > 100\right) \times 11 = e^{-100/515} \times 11 = 9.05865$$

Solution 12.12

The claim payment in Year 2004 is $X - 100 \mid X > 100$, which follows the same distribution as *X*, exponential with mean 500. The claim payment in Year 2005 is $X_{05} - 100 \mid X_{05} > 100$, which follows the same distribution as X_{05}, exponential with mean 515.

Solution 12.13

In Solution 12.6 we calculated the expected annual claims payments in Year 2004 as:

$$E[S] = \lambda E\left[(X - 100)_+\right] = 4{,}093.65$$

If there is an ordinary deductible of *d* per loss, then the expected annual claims payments in Year 2005 are:

$$E[S_{05}] = E[N_{05:L}]E\left[(X_{05} - d)_+\right] = \underbrace{E[N_{05:L}]}_{\text{Solution 12.10}} \times \underbrace{E\left[X_{05} - d \mid X_{05} > d\right]}_{\substack{\text{Same mean as } X_{05} \text{ since} \\ \text{amounts are exponentially} \\ \text{distributed}}} \times \Pr\left(X_{05} > d\right)$$

$$= 11 \times 515 \times e^{-d/515}$$

Equating these expected values and solving for *d* results in $d = 167.31$

Solution 12.14

The payment per loss in Year 2005 would be:

$$Y_{05} = \begin{cases} 0 & \text{if } X_{05} \le 100 \\ X_{05} - 100 & \text{if } 100 < X_{05} < 100 + L \\ L & \text{if } 100 + L \le X_{05} \end{cases}$$

$$= X_{05} \wedge (100 + L) \; - \; X_{05} \wedge 100$$

Using the expected limited loss formula for an exponential distribution, we have:

$$E\left[X_{05} \wedge x\right] = \theta_{05}\left(1 - e^{-x/\theta_{05}}\right) = 515\left(1 - e^{-x/515}\right) \;\Rightarrow$$

$$E\left[Y_{05}\right] = E\left[X_{05} \wedge (100+L)\right] - E\left[X_{05} \wedge 100\right]$$

$$= 515\left(1 - e^{-(100+L)/515}\right) - 515\left(1 - e^{-100/515}\right)$$

$$= 424.10970\left(1 - e^{-L/515}\right)$$

To hold expected annual claims payments in Year 2005 to the same level as in Year 2004, we would have:

$$4{,}093.65 = E\left[S_{04}\right] = E\left[S_{05}\right] = E\left[N_{05:L}\right] E\left[Y_{05}\right] = 11 \times 424.10970\left(1 - e^{-L/515}\right)$$

$$\Rightarrow \; e^{-L/515} = 0.12251 \;\Rightarrow\; L = 1{,}081.25$$

Solution 12.15

In Solution 12.6 when it was assumed that there was an ordinary deductible of 100 per loss, we calculated the expected value and variance of the annual claims payments as:

$$E[S] = \lambda E\left[(X-100)_+\right] = 4{,}093.65 \quad , \quad \operatorname{var}(S) = \lambda E\left[(X-100)_+^2\right] = 4{,}093{,}653.77$$

If the deductible amount is replaced by a policy limit that results in the same level of expected annual claims, we have:

$$409.36538 = E\left[(X-100)_+\right] = E\left[X \wedge L\right] = 500\left(1 - e^{-L/500}\right) \;\Rightarrow\; L = 853.88590$$

With this policy limit, the variance in annual claims payments is determined as follows:

$$E\left[(X \wedge L)^2\right] = 2\theta^2\,\Gamma(3\,;\,L/\theta) + L^2\ e^{-L/\theta} \qquad \text{(Table 10.4)}$$

$$= 500{,}000\left(1 - e^{-1.70777}\left(1 + 1.70777 + \frac{1.70777^2}{2!}\right)\right) + 853.88590^2\ e^{-1.70777}$$

$$= 122{,}414.8843\ +\ 132{,}167.2383 = 254{,}582.1226$$

$$\Rightarrow \quad \text{var}(S) = \lambda\,E\left[(X \wedge L)^2\right] = 10 \times 254{,}582.1226 = 2{,}545{,}821.23$$

This is 62.2% of the variance with a deductible rather than a limit.

Solution 12.16

The distribution of $Y = (X-1)_+$ is:

$$\Pr(Y=0) = 0.7 \quad , \quad \Pr(Y=1) = 0.2 \quad , \quad \Pr(Y=2) = 0.1$$

Since *N* is Poisson distributed with mean $\lambda = 4$, the recursion formula and the starting value (Section 11.3) are:

$$\Pr(S=0) = P_N\left(\Pr(Y=0)\right) = e^{\lambda(\Pr(Y=0)-1)} = e^{4(0.7-1)} = e^{-1.2}$$

$$\lambda_i = \lambda\Pr(Y=i) \Rightarrow \lambda_1 = 0.8\ ,\ \lambda_2 = 0.4 \Rightarrow$$

$$\Pr(S=n) = \frac{1}{n}\left(1\cdot\lambda_1\cdot\Pr(S=n-1) + 2\cdot\lambda_2\cdot\Pr(S=n-2)\right)$$

$$= \frac{1}{n}\left(0.8\Pr(S=n-1) + 0.8\ \Pr(S=n-2)\right)$$

Use this formula successively with $n=1$, 2, and 3:

$$\Pr(S=1)\ =\ 0.8\Pr(S=0) = 0.8\,e^{-1.2}$$

$$\Pr(S=2)\ =\ \frac{1}{2}\left(0.8\Pr(S=1) + 0.8\Pr(S=0)\right) = 0.72e^{-1.2}$$

$$\Pr(S=3)\ =\ \frac{1}{3}\left(0.8\Pr(S=2) + 0.8\ \Pr(S=1)\right)\ = 0.40533\,e^{-1.2}$$

Totaling these 4 probabilities you will find that: $\Pr(S \le 3) = 2.92533\,e^{-1.2}\ = 0.88109$.

Solution 12.17

In order to calculate from combinatorial reasoning, we must first filter out the zero terms in the sum and then adjust the frequency distribution (see Section 11.7). Let's set notation:

$$
\begin{aligned}
S &= Y_1 + \cdots + Y_{N_L} \quad \text{where } N_L \sim \text{Poisson } \lambda = 4 \\
&= Z_1 + \cdots + Z_{N_P} \quad \text{where } Z = Y|Y>0 \text{ and } N_P \sim \text{Poisson } \lambda^* = \lambda \Pr(Y>0)
\end{aligned}
$$

From Solution 12.16, we have:

$$\Pr(Z=1) = \frac{\Pr(Y=1)}{\Pr(Y>0)} = \frac{2}{3} \quad , \quad \Pr(Z=2) = \frac{\Pr(Y=2)}{\Pr(Y>0)} = \frac{1}{3} \quad , \quad \lambda^* = 4 \times 0.3 = 1.2$$

Now it is easy to duplicate the probability calculations in Solution 12.16. For example, we have:

$$\Pr(S=0) = \Pr(N_P=0) = e^{-\lambda^*} = e^{-1.2}$$

$$\Pr(S=1) = \Pr(N_P=1 \text{ and } Z=1) = e^{-1.2}\,\frac{1.2^1}{1!} \times \frac{2}{3} = 0.8\,e^{-1.2}$$

It is left to the reader to verify the calculations for $\Pr(S=2)$ and $\Pr(S=3)$.

Solution 12.18

$$
\begin{aligned}
E\left[(S-3.8)_+\right] &= E[S] - E[S \wedge 3.8] \\
&= \lambda E\left[(X-1)_+\right] - \left(0\cdot\Pr(S=0) + 1\cdot\Pr(S=1) + 2\cdot\Pr(S=2) + 3\cdot\Pr(S=3) + 3.8\times\Pr(S\ge 4)\right) \\
&= 4\times 0.4 - \left(0.8e^{-1.2} + 2\times 0.72e^{-1.2} + 3\times 0.40533e^{-1.2} + 3.8\times 0.11891\right) = 0.10723
\end{aligned}
$$

Solution 12.19

Aggregate annual claims follow a compound Poisson distribution:

$$S = (X_1 - 100)_+ + \cdots + (X_{N_L} - 100)_+ \quad \text{where } N_L \sim \text{Poisson } \lambda = 10$$

The severity model X follows an exponential distribution with mean $\theta = 500$. For each loss $Y = (X-100)_+$ of the insurer, the reinsurer pays the insurer an amount R equal to the excess of Y over 500, if there is an excess:

$$R = (Y - 500)_+ = \left((X-100)_+ - 500\right)_+ = (X-600)_+$$

The total payment by the reinsurer to the insurer is thus $S_{\text{re}} = R_1 + \cdots + R_{N_L}$. The pure reinsurance premium is the expected value of S_{re}. Since X follows an exponential distribution with mean $\theta = 500$, we know that $X - 600 \mid X > 600$ follows this same exponential distribution. As a result, we have:

$$E[R] = E\left[(X-600)_+\right] = E\left[X - 600 \mid X > 600\right] \Pr\left(X > 600\right) = 500 \times e^{-600/500} = 150.59711$$

$$E\left[S_{\text{re}}\right] = E\left[N_L\right] E[R] = 10 \times 150.59711 = 1{,}505.97$$

Solution 12.20

Aggregate annual claims follow a compound Poisson distribution:

$$S = \left(X_1 - 100\right)_+ + \cdots + \left(X_{N_L} - 100\right)_+ \quad \text{where } N_L \sim \text{Poisson } \lambda = 10$$

The severity model X follows an exponential distribution with mean $\theta = 500$. For each loss $Y = \left(X - 100\right)_+$ of the insurer, the reinsurer pays the insurer an amount $R = 0.25\,Y$. In this case it is easy to see that the total reinsurance payments are $S_{\text{Re}} = 0.25\,S$. So the pure reinsurance premium is:

$$E\left[S_{\text{Re}}\right] = 0.25\,E[S] = 0.25 \times \underbrace{4{,}093.65377}_{\text{Solution 12.6}} = 1{,}023.41$$