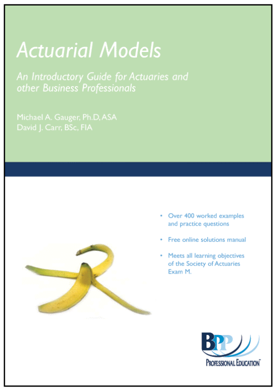

Actuarial models

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Solutions to practice questions – Chapter 13

Solution 13.1

The rate of this process is $\lambda = 0.2$ per minute. So the number of coins found during a 2-minute walk follows a Poisson distribution with mean $2\lambda = 0.40$. As a result, we have:

$$\Pr\left(N(2) \geq 2\right) = 1 - e^{-0.4}\left(1 + 0.4\right) = 0.06155$$

Solution 13.2

$$N(2) \sim \text{Poisson mean } 0.4 \quad \Rightarrow \quad E[N] = \text{var}(N) = 0.4$$

Solution 13.3

The distribution of $N(5)$ is Poisson with parameter $5\lambda = 1$.

$$\{N(5) \geq 2 \text{ and } N(10) \geq 3\} = \{N(5) = 2 \text{ and } N(10) - N(5) \geq 1\} \cup \{N(5) \geq 3\}$$

$$\Pr\left(N(5) \geq 2 \text{ and } N(10) \geq 3\right) = \Pr\left(N(5) = 2 \text{ and } N(10) - N(5) \geq 1\right) + \Pr\left(N(5) \geq 3\right)$$

$$= \Pr\left(N(5) = 2\right) \underbrace{\Pr\left(N(10) - N(5) \geq 1\right)}_{\text{same as } \Pr(N(5) \geq 1)} + \Pr\left(N(5) \geq 3\right)$$

$$= e^{-1}\frac{1^2}{2!} \times \left(1 - e^{-1}\right) + \left(1 - e^{-1}\left(1 + 1 + \frac{1^2}{2!}\right)\right) = 0.19657$$

Solution 13.4

$$\Pr\left(N(5)=2 \mid N(10)=3\right) = \frac{\Pr\left(N(5)=2 \text{ and } N(10)=3\right)}{\Pr\left(N(10)=3\right)}$$

$$= \frac{\Pr\left(N(5)=2 \text{ and } N(10)-N(5)=1\right)}{\Pr\left(N(10)=3\right)} = \frac{\Pr\left(N(5)=2\right)\Pr\left(N(10)-N(5)=1\right)}{\Pr\left(N(10)=3\right)}$$

$$= \frac{\Pr\left(N(5)=2\right)\Pr\left(N(5)=1\right)}{\Pr\left(N(10)=3\right)} = \frac{e^{-1}\dfrac{1^2}{2!} \times e^{-1}\dfrac{1^1}{1!}}{e^{-2}\dfrac{2^3}{3!}} = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{3}{8}$$

Solution 13.5

Since the process rate is $\lambda = 0.4$ per day, the inter-arrival time for this process, *T*, is exponentially distributed with mean $1/\lambda = 2.5$ days. You are asked to calculate $\Pr\left(T \le 7 \mid T > 5\right)$. Due to the memory-less property of the exponential distribution, we have:

$$\Pr\left(T \le 7 \mid T > 5\right) = \Pr\left(T - 5 \le 2 \mid T > 5\right) = \Pr\left(T \le 2\right) = \int_0^2 f_T(t)\,dt = \int_0^2 0.4e^{-0.4t}\,dt = 1 - e^{-0.8} = 0.55067$$

Solution 13.6

We must solve the inequality:

$$0.90 \le \Pr\left(N(n) \ge 2\right) = 1 - e^{-0.4n}\left(1 + 0.4n\right)$$

By trial and error you will find that the right hand side is 0.874 when $n = 9$, and it is 0.908 when $n = 10$. So the intersection must be observed for 10 full days to have at least a 90% chance of seeing at least 2 accidents.

Solution 13.7

Since $\lambda = 0.4$, the time of the third event, S_3, follows a gamma distribution with $\alpha = 3$ and $\theta = 1/\lambda = 2.5$. The mean and variance of this event time are:

$$E\left[S_3\right] = \alpha\theta = 7.5 \qquad \operatorname{var}\left(S_3\right) = \alpha\theta^2 = 18.75$$

Solution 13.8

Using the results in Solution 13.7, we see that we must calculate the exact probability of the event:

$$3.16987 = 7.5 - \sqrt{18.75} \le S_3 \le 7.5 + \sqrt{18.75} = 11.83013$$

From Theorem 13.2 (iv), we have the following formula for the cdf of S_3:

$$F_{S_3}(t) = \Pr(S_3 \le t) = \Pr(N(t) \ge 3) = 1 - e^{-0.4t}\left(1 + 0.4t + \frac{(0.4t)^2}{2!}\right)$$

$$= 1 - e^{-0.4t}\left(1 + 0.4t + 0.08t^2\right)$$

So we have:

$$\Pr(3.16987 \le S_3 \le 11.83013) = F_{S_3}(11.83013) - F_{S_3}(3.16987) = 0.85089 - 0.13557 = 0.71532$$

Solution 13.9

We must thin the process producing losses to losses exceeding the limit $L = 500$. The probability that a loss exceeds 500 is:

$$\Pr(X > 500) = s_X(500) = \left(\frac{\theta}{\theta + 500}\right)^{\alpha} = \left(\frac{250}{750}\right)^2 = \frac{1}{9}$$

So the Poisson process $N_1(t)$ counting losses in excess of 500 has rate $\lambda_1 = \lambda\, p_1 = 10\Pr(X > 500) = 10/9$ per month. We are asked to calculate $E\left[S_4^{(1)}\right]$ and $\operatorname{var}\left(S_4^{(1)}\right)$ for this thinned process:

$$S_4^{(1)} \sim \text{gamma } \alpha = 4,\ \theta_1 = 1/\lambda_1 = 9/10 \Rightarrow E\left[S_4^{(1)}\right] = \alpha\theta_1 = 3.6 ,\ \operatorname{var}\left(S_4^{(1)}\right) = \alpha\theta_1^2 = 3.24$$

Solution 13.10

$$\Pr\left(S_3^{(1)} \le 1\right) = \Pr(N_1(1) \ge 3) = 1 - e^{-\lambda_1}\left(1 + \lambda_1 + \frac{\lambda_1^2}{2!}\right) = 0.10183$$

Solution 13.11

Here we are asked to calculate the expected value and variance of the inter-arrival time for the thinned process:

$$T^{(1)} \sim \text{exponential } \theta_1 = 1/\lambda_1 = 9/10 \Rightarrow E\left[T^{(1)}\right] = \theta_1 = 9/10 ,\ \operatorname{var}\left(T^{(1)}\right) = \theta_1^2 = 81/100$$

Solution 13.12

We need to apply the reasoning employed in the solution to Example 13.9. The method of this solution was extended to give a general formula in Theorem 13.3.

We have two categories of losses:

- C_1 is the category of losses greater than 500, and $p_1 = \Pr(X > 500) = 1/9$
- C_2 is the category of losses less than or equal to 500, and $p_2 = 1 - p_1 = 8/9$

The probability that two losses in excess of the limit occur before five losses at or below the limit is the same as the probability of 2 or more successes in the next 6 trials (*ie* a trial consists of waiting for the next loss) where the probability of success is $p = p_1 = 1/9$ ("success" means that the loss exceeds 500). Using the binomial probability function, we have:

$$\Pr(\text{at least 2 successes in 6 trials}) = 1 - \binom{6}{0}\left(\frac{1}{9}\right)^0\left(\frac{8}{9}\right)^6 - \binom{6}{1}\left(\frac{1}{9}\right)^1\left(\frac{8}{9}\right)^5 = 0.13678$$

Solution 13.13

Since the rate function is $\lambda(t) = 100 - 10t$ for $0 \le t \le 10$, the mean value function is:

$$m(t) = \int_0^t \lambda(s)\,ds = \int_0^t 100 - 10s\,ds = 100t - 5t^2 \quad \text{for} \quad 0 \le t \le 10$$

In general, we have $N(t) \sim$ Poisson $m(t)$. So $E[N(10)] = m(10) = 500$.

Solution 13.14

The distribution of $N(5)$ is Poisson with parameter equal to $m(5) = 375$. Making a normal approximation to the distribution of $N(5)$ with $\mu = E[N_5] = 375$ and $\sigma^2 = \text{var}(N_5) = 375$, we have:

$$\Pr(N(5) > 425) \approx \Pr\left(N(0,1) > \frac{425 - 375}{\sqrt{375}}\right) = 1 - \Phi(2.582) \approx 0.005$$

Solution 13.15

$$f_{T_1}(t) = \lambda(t)\,e^{-m(t)} = (100 - 10t)\,e^{-(100t - 5t^2)} \quad \text{for} \quad t > 0$$

Solution 13.16

Here we have time-dependent thinning. Category 1 corresponds to express trains and category 2 corresponds to local trains. We are given that:

$$p_1(t) = \begin{cases} 0.50 & \text{for } 1 < t \le 3 \\ 0.20 & \text{for other times of day} \end{cases} \quad , \quad p_2(t) = \begin{cases} 0.50 & \text{for } 1 < t \le 3 \\ 0.80 & \text{for other times of day} \end{cases}$$

where time is measured in hours from 5 am.

We are asked to calculate $E\left[N_1(4)\right]$. According to the discussion in Section 13.6, we have:

$$N_1(t) \sim \text{Poisson with parameter } m_1(t) = \lambda \int_0^t p_1(s)\,ds$$

$$m_1(4) = \lambda \int_0^4 p_1(s)\,ds = 5\left(\int_0^1 0.2\,ds + \int_1^3 0.5\,ds + \int_3^4 0.2\,ds\right) = 7$$

Solution 13.17

We are asked to calculate $\Pr\left(N_1(2.5) - N_1(1.5) \ge 2\right)$. We must first calculate the mean value function for this thinned process:

$$m_1(2.5) - m_1(1.5) = \int_{1.5}^{2.5} \lambda_1(t)\,dt = \lambda \int_{1.5}^{2.5} p_1(t)\,dt \; = \; 5\int_{1.5}^{2.5} 0.5\,dt \; = \; 2.5$$

As a result, we know that $N_1(2.5) - N_1(1.5)$ is Poisson distributed with parameter 2.5. Therefore:

$$\Pr\left(N_1(2.5) - N_1(1.5) \ge 2\right) = 1 - e^{-2.5}(1 + 2.5) = 0.71270$$

Solution 13.18

Think of resetting time 0 to 7am. So the rate function for the express train process is thus:

$$\lambda_1(t) = 5p_1(t) = \begin{cases} 2.5 & \text{for } 0 \le t \le 1 \\ 1.0 & \text{for } 1 < t \le 16 \end{cases} \quad \text{(note: service stops at 11 pm, time 16)}$$

The mean value function is:

$$m_1(t) = \int_0^t \lambda_1(s)\,ds \; = \begin{cases} 2.5t & \text{for } 0 \le t \le 1 \\ 2.5 + (t-1) & \text{for } 1 \le t \le 16 \end{cases}$$

The survival function for $T_1^{(1)}$ is:

$$\Pr\left(T_1^{(1)} > t\right) = \Pr\left(N_1(t) = 0\right) = e^{-m_1(t)}$$

If time $t=0$ corresponds to 7 am, then 7:06 am is time $t=0.1$ and 7:30 am is time $t=0.5$. We are asked to calculate the conditional probability:

$$\Pr\left(T_1^{(1)} \le 0.5 \mid T_1^{(1)} > 0.1\right) = 1 - \Pr\left(T_1^{(1)} > 0.5 \mid T_1^{(1)} > 0.1\right) = 1 - \frac{\Pr\left(T_1^{(1)} > 0.5\right)}{\Pr\left(T_1^{(1)} > 0.1\right)}$$

$$= 1 - \frac{e^{-m_1(0.5)}}{e^{-m_1(0.1)}} = 1 - e^{-\left(m_1(0.5) - m_1(0.1)\right)} = 1 - e^{-(1.25 - 0.25)} = 0.63212$$

Solution 13.19

We are given $\lambda = 2$ per minute, and: $\Pr(X = 0.85) = 0.8$, $\Pr(X = 1.25) = 0.2$. So total sales in t minutes is modeled by the compound Poisson process:

$$S(t) = X_1 + \cdots + X_{N(t)} \quad \text{where} \quad N(t) \sim \text{Poisson } 2t$$

It is easy to verify that $E[X] = 0.93000$, $E\left[X^2\right] = 0.89050$. Therefore we have:

$$E\left[S(60)\right] = 60\lambda E[X] = (2\times 60)\times 0.93 = 111.60 \ , \ \operatorname{var}\left(S(60)\right) = 60\lambda E\left[X^2\right] = (2\times 60)\times 0.89050 = 106.86$$

Solution 13.20

Split the Poisson process into the sum of the processes corresponding to small coffee purchases and large coffee purchases:

$$N(t) = N_s(t) + N_l(t) \quad \text{where} \quad \lambda_s = \lambda\times 0.8 = 1.6 \ , \ \lambda_l = \lambda\times 0.2 = 0.4$$

We are asked to compute $E\left[S(60) \mid N_s(60) = 120\right]$ and $\operatorname{var}\left(S(60) \mid N_s(60) = 120\right)$. So write aggregate sales in terms of the frequencies of large and small coffee purchases:

$$S(60) = 0.85 N_s(60) + 1.25 N_l(60)$$

Now use the independence of $N_s(60)$ and $N_l(60)$:

$$\begin{aligned} E\left[S(60) \mid N_s(60) = 120\right] &= E\left[0.85 N_s(60) + 1.25 N_l(60) \mid N_s(60) = 120\right] \\ &= E\left[102 + 1.25 N_l(60) \mid N_s(60) = 120\right] = E\left[102 + 1.25 N_l(60)\right] \\ &= 102 + 1.25\times(0.4\times 60) = 132 \end{aligned}$$

$$\begin{aligned} \operatorname{var}\left(S(60) \mid N_s(60) = 120\right) &= \operatorname{var}\left(0.85 N_s(60) + 1.25 N_l(60) \mid N_s(60) = 120\right) \\ &= \operatorname{var}\left(102 + 1.25 N_l(60) \mid N_s(60) = 120\right) = \operatorname{var}\left(102 + 1.25 N_l(60)\right) \\ &= 1.25^2 \times (0.4\times 60) = 37.50 \end{aligned}$$

