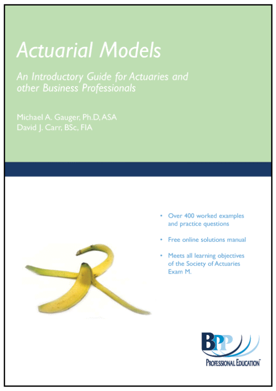

PROFESSIONAL EDUCATION

Actuarial models

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Published by BPP Professional Education

Solutions to practice questions – Chapter 14

Solution 14.1

We have:

$$\mathbf{Q}_{50} = \begin{pmatrix} p_{50} & q_{50} \\ 0 & 1 \end{pmatrix} \qquad {}_2\mathbf{Q}_{50} = \begin{pmatrix} {}_2p_{50} & {}_2q_{50} \\ 0 & 1 \end{pmatrix}$$

Solution 14.2

With de Moivre's law we have $l_x = 90 - x$ for $0 \le x \le 90$. So we have:

$$Q_{50}^{(1,1)} = p_{50} = \frac{l_{51}}{l_{50}} = \frac{39}{40} \qquad Q_{50}^{(1,2)} = q_{50} = \frac{1}{40}$$

$$Q_{51}^{(1,1)} = p_{51} = \frac{l_{52}}{l_{51}} = \frac{38}{39} \qquad Q_{51}^{(1,2)} = q_{51} = \frac{1}{39}$$

Solution 14.3

The transition probability ${}_kQ_n^{(1,1)}\,Q_{n+k}^{(1,2)}$ is the conditional probability of transition from state 1 to state 2 during the time period $(n+k, n+k+1]$, given state 1 at time n. In survival model notation this probability is ${}_{k|}q_n$.
When $n = 50$ and $k = 10$, and mortality follows de Moivre's law with $\omega = 90$, we have:

$${}_{10}Q_{50}^{(1,1)}\,Q_{60}^{(1,2)} = {}_{10|}q_{50} = {}_{10}p_{50}\;q_{60} = \frac{30}{40} \times \frac{1}{30} = \frac{1}{40}$$

Solution 14.4

Since the third policy year starts at time 2, we are asked to calculate ${}_2Q_0^{(2,1)} + {}_2Q_0^{(2,2)}$. These two numbers being summed are the $(2,1)$ and $(2,2)$ entries of the matrix ${}_2\mathbf{Q}_0 = \mathbf{Q}_0\,\mathbf{Q}_1$. Using matrix multiplication we have:

$$ {}_2Q_0^{(2,1)} = (0.20 \times 0.90) + (0.70 \times 0.10) + (0.05 \times 0) + (0.05 \times 0) = 0.25 $$
$$ {}_2Q_0^{(2,2)} = (0.20 \times 0.05) + (0.70 \times 0.80) + (0.05 \times 0.10) + (0.05 \times 0) = 0.575 $$

So the total probability is 0.825.

Solution 14.5

We have standard at time 0, standard at time 1, and then preferred at time 2, the start of the third policy year. The probability of this sequence of ratings is:

$$ Q_0^{(2,2)}\,Q_1^{(2,1)} = 0.70 \times 0.10 = 0.07 $$

Solution 14.6

We are asked to compute ${}_2Q_0^{(3,4)}$, the $(3,4)$ entry of ${}_2\mathbf{Q}_0 = \mathbf{Q}_0\,\mathbf{Q}_1$:

$$ {}_2Q_0^{(3,4)} = (0 \times 0.02) + (0.10 \times 0.05) + (0.60 \times 0.50) + (0.30 \times 1) = 0.605 $$

Solution 14.7

The sequence of rating is preferred at times 0, 1, and 2, and then standard at time 3, the start of the fourth policy year. The probability of this sequence of ratings is:

$$ P_0^{(1)}\,P_1^{(1)}\,Q_2^{(1,2)} = Q_0^{(1,1)}\,Q_1^{(1,1)}\,Q_2^{(1,2)} = 0.85 \times 0.90 \times 0.05 = 0.03825 $$

Solution 14.8

The sequence of ratings is preferred at times 0 and 2, and then standard at time 3, the start of the fourth policy year. The probability of this sequence of ratings is:

$$ {}_2Q_0^{(1,1)}\,Q_2^{(1,2)} = \underbrace{\big((0.85 \times 0.90) + (0.10 \times 0.10) + (0.03 \times 0) + (0.02 \times 0)\big)}_{\text{the } (1,1) \text{ entry of } \mathbf{Q}_0\mathbf{Q}_1} \times 0.05 = 0.03875 $$

Solution 14.9

The unconditional distribution at time 0 is $\vec{\boldsymbol{\pi}}_0 = \begin{pmatrix}0.10 & 0.80 & 0.10 & 0\end{pmatrix}$. The unconditional distribution at time 2, the start of calendar year 2007, is:

$$\vec{\boldsymbol{\pi}}_2 = \vec{\boldsymbol{\pi}}_0\,\mathbf{Q}_0\,\mathbf{Q}_1 = \begin{pmatrix}0.10 & 0.80 & 0.10 & 0\end{pmatrix}\begin{pmatrix}0.7750 & 0.1255 & 0.0425 & 0.0570 \\ 0.2500 & 0.5750 & 0.0610 & 0.1140 \\ 0.0100 & 0.1400 & 0.2450 & 0.6050 \\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}0.27850 & 0.48655 & 0.07755 & 0.15740\end{pmatrix}$$

Now multiply this vector of probabilities by the 400 new policies at time 0 and you have the expected breakdown of policyholders from this group at the start of calendar year 2007:

$$400\,\vec{\boldsymbol{\pi}}_2 = \begin{pmatrix}111.40 & 194.62 & 31.02 & 62.96\end{pmatrix}$$

Solution 14.10

This problem is not concerned with cash flow while in a state or cash flow upon transition. However, it still makes sense to use the Markov chain probabilities. Since the driver is initially rated standard, there is an expected loss of 1,000 at time 1 that must be discounted at 4%.

Now consider the start of year 2006:

- There is an expected loss of 750 at time 2 with probability $Q_0^{(2,1)} = 0.20$ if the driver is rated preferred for year 2006
- There is an expected loss of 1,000 at time 2 with probability $Q_0^{(2,2)} = 0.70$ if the driver is rated standard for year 2006
- There is an expected loss of 2,000 at time 2 with probability $Q_0^{(2,3)} = 0.05$ if the driver is rated sub-standard for year 2006

So the actuarial present value of the insurer's losses on a standard issue in the first two calendar years is:

$$\text{APV} = \left(1{,}000 \times 1.04^{-1}\right) + \left(750 \times 1.04^{-2} \times Q_0^{(2,1)}\right) + \left(1{,}000 \times 1.04^{-2} \times Q_0^{(2,2)}\right) + \left(2{,}000 \times 1.04^{-2} \times Q_0^{(2,3)}\right)$$
$$= 1{,}839.87$$

Solution 14.11

This actuarial present value of the premiums for years 2005 and 2006 is:

$$\text{APV} = P + \left((P-50) \times 1.04^{-1} \times \underbrace{Q_0^{(2,1)}}_{0.20}\right) + \left(P \times 1.04^{-1} \times \underbrace{Q_0^{(2,2)}}_{0.70}\right) + \left(2P \times 1.04^{-1} \times \underbrace{Q_0^{(2,3)}}_{0.05}\right)$$
$$= 1.96154\,P - 9.61538$$

Solution 14.12

We must equate the actuarial present value of losses and premiums for the 2-year period:

$$1.96154\,P - 9.61538 = \text{APV(premium)} = \text{APV}\left(\text{losses}\right) = 1{,}839.87$$
$$\Rightarrow \quad P = 942.87$$

Solution 14.13

The expected loss for this policyholder at the end of year 2006 is 2,000. The premium to be paid at the start of year 2006 is $2P$. So the terminal reserve at the end of the first year is:

$$2{,}000 \times 1.04^{-1} - 2P = 37.33$$

Solution 14.14

The distribution at time 0 is $\begin{pmatrix}1 & 0\end{pmatrix}$ since you are given state 1 at time 0. So the distribution at time 1 is the first row of the matrix $\mathbf{Q}_0$: $\vec{\boldsymbol{\pi}}_1 = \begin{pmatrix}0.90 & 0.10\end{pmatrix}$. The distribution at time 2 is:

$$\vec{\boldsymbol{\pi}}_2 = \vec{\boldsymbol{\pi}}_1\,\mathbf{Q}_1 = \begin{pmatrix}0.90 & 0.10\end{pmatrix}\begin{pmatrix}0.80 & 0.20\\ 0.70 & 0.30\end{pmatrix} = \begin{pmatrix}0.79 & 0.21\end{pmatrix}$$

Solution 14.15

The question asks for the value of ${}_2Q_0^{(1,1)}\,Q_2^{(1,2)} = \left(0.90 \times 0.80 + 0.10 \times 0.70\right) \times 0.20 = 0.1580$

Solution 14.16

From the given information we have $P_0^{(1)} = 0.90$ and $P_k^{(1)} = 0.80$ for $k \geq 1$. So we have:

$$\Pr\left(N = 1\right) = \left(1 - P_0^{(1)}\right) = 0.10$$
$$\Pr\left(N = k\right) = P_0^{(1)} P_1^{(1)} \cdots P_{k-2}^{(1)}\left(1 - P_{k-1}^{(1)}\right) = 0.90 \times 0.80^{k-2} \times 0.20 \quad \text{for } k \geq 2$$

Solution 14.17

$$\text{APV} = \left(50 \times 1.01^{-0} \times 0\right) + \left(50 \times 1.01^{-1} \times Q_0^{(1,2)}\right) + \left(50 \times 1.01^{-2} \times {}_2Q_0^{(1,2)}\right) = 15.24$$

since: $Q_0^{(1,2)} = 0.10$, ${}_2Q_0^{(1,2)} = 0.90 \times 0.20 + 0.10 \times 0.30 = 0.21$

Solution 14.18

$$\text{APV} = \left(200 \times 1.01^{-1} \times Q_0^{(1,2)}\right) + \left(200 \times 1.01^{-2} \times Q_0^{(1,1)} Q_1^{(1,2)}\right) + \left(200 \times 1.01^{-3} \times {}_2Q_0^{(1,1)}\ Q_2^{(1,2)}\right)$$
$$= 19.80 + 35.29 + 30.67 = 85.76$$

Solution 14.19

The value of three premiums is $P\left(1+1.01^{-1}+1.01^{-2}\right) = 2.97040P$. Setting the value of the premiums equal to the actuarial present value of losses results in:

$$2.97040P = (15.24 + 85.76) \Rightarrow \quad P = 34.00$$

Solution 14.20

At the end of the first period, the APV of future losses during the next two periods is:

$$\text{APV} = \left(50 \times 1.01^{-0} \times 0\right) + \left(50 \times 1.01^{-1} \times 0.20\right) + \left(200 \times 1.01^{-1} \times 0.20\right) + \left(200 \times 1.01^{-2} \times 0.80 \times 0.20\right)$$
$$= 9.90 + 39.60 + 31.37 = 80.87$$

At the end of the first period, the APV of future premiums during the next two periods is:

$$\text{APV} = 34 + 34 \times 1.01^{-1} = 67.66$$

So the reserve after 1 period is:

$${}_1V = 80.87 - 67.66 = 13.21$$