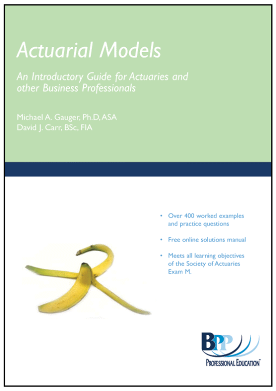

Actuarial models

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Solutions to practice questions – Chapter 7

Solution 7.1

Survival for a period of time means remaining a member of the group for this period, in other words, avoiding all the modes of decrement.

Solution 7.2

(i) $${}_2p_{26}^{(\tau)} = \frac{l_{28}^{(\tau)}}{l_{26}^{(\tau)}} = \frac{919}{950}$$

(ii) $${}_2q_{25}^{(\tau)} = \frac{d_{25}^{(\tau)} + d_{26}^{(\tau)}}{l_{25}^{(\tau)}} = \frac{50+14}{1{,}000}$$

(iii) $${}_{1|}q_{26}^{(2)} = \frac{d_{27}^{(2)}}{l_{26}^{(\tau)}} = \frac{8}{950}$$

(iv) $${}_2q_{26}^{(1)} = \frac{d_{26}^{(1)} + d_{27}^{(1)}}{l_{26}^{(\tau)}} = \frac{8+9}{950}$$

Solution 7.3

From the formulas for the force functions, we have:

$$\mu^{(i)}(x+t) = \frac{r_i}{20-t} \quad \Rightarrow \quad {}_tp_x'^{(i)} = \exp\left(-\int_0^t \frac{r_i}{20-s}\,ds\right) = \left(\frac{20-t}{20}\right)^{r_i} \quad \text{for } 0 \le t < 20$$

Cause 1 , $r_1 = 1$

$${}_tp_x'^{(1)} = \frac{20-t}{20} \quad , \quad f_{T_1}(t) = {}_tp_x'^{(1)}\,\mu^{(1)}(x+t) = \frac{20-t}{20} \times \frac{1}{20-t} = \frac{1}{20} \quad \text{and}$$

$${}_tq_x'^{(1)} = F_{T_1}(t) = \frac{t}{20} \quad \text{for } 0 \le t < 20$$

Cause 2 , r_2 = 0.5

$${}_tp_x'^{(2)} = \left(\frac{20-t}{20}\right)^{0.5} \quad , \quad f_{T_2}(t) = {}_tp_x'^{(2)}\,\mu^{(2)}(x+t) = \left(\frac{20-t}{20}\right)^{0.5} \times \frac{0.5}{20-t} = \frac{1}{\sqrt{80(20-t)}} \quad \text{and}$$

$${}_tq_x'^{(2)} = F_{T_2}(t) = 1 - \left(\frac{20-t}{20}\right)^{0.5} \quad \text{for } 0 \le t < 20$$

Solution 7.4

$$q_x'^{(1)} = \frac{1}{20} = 0.05000 \quad , \quad q_x'^{(2)} = 1 - \left(\frac{20-1}{20}\right)^{0.5} = 0.02532$$

Solution 7.5

The joint density function is not a probability, but $f_{T,J}(t,j)dt$ is approximately the probability that life (x) departs the group between times t and $t+dt$ as a result of cause j.

Solution 7.6

The waiting time variables $T_1, \ldots, T_r$ are assumed to be independent. Since T is the minimum of these waiting times, the event $T > t$ is the intersection of the independent events $T_i > t$. As a result, we have:

$${}_tp_x^{(\tau)} = \Pr(T > t) = \Pr(T_1 > t) \cdots \Pr(T_r > t) = {}_tp_x'^{(1)} \cdots {}_tp_x'^{(r)}$$

So for the pair of forces in Question 7.3, we have:

$${}_tp_x^{(\tau)} = \left(\frac{20-t}{20}\right) \times \left(\frac{20-t}{20}\right)^{0.5} = \left(\frac{20-t}{20}\right)^{1.5} \quad \text{for } 0 \le t \le 20$$

Solution 7.7

$$f_{T,J}(t,j) = {}_tp_x^{(\tau)}\,\mu^{(j)}(x+t) \Rightarrow$$

$$f_{T,J}(t,1) = {}_tp_x^{(\tau)}\,\mu^{(1)}(x+t) = \left(\frac{20-t}{20}\right)^{1.5} \frac{1}{20-t} = \frac{(20-t)^{0.5}}{20^{1.5}} \quad \text{for } 0 \le t \le 20$$

$$f_{T,J}(t,2) = {}_tp_x^{(\tau)}\,\mu^{(2)}(x+t) = \left(\frac{20-t}{20}\right)^{1.5} \frac{0.5}{20-t} = \frac{(20-t)^{0.5}}{2 \times 20^{1.5}} \quad \text{for } 0 \le t \le 20$$

Solution 7.8

$$\int_0^{20} f_{T,J}(t,1)\,dt + \int_0^{20} f_{T,J}(t,2)\,dt = \int_0^{20} \frac{1.5(20-t)^{0.5}}{20^{1.5}}\,dt = -\frac{(20-t)^{1.5}}{20^{1.5}}\Bigg|_0^{20} = 0 + 1 = 1$$

Solution 7.9

$$q_x^{(1)} = \Pr(T \le 1\ ,\ J = 1) = \int_0^1 f_{T,J}(t,1)\,dt$$

$$= \int_0^1 \frac{(20-t)^{0.5}}{20^{1.5}}\,dt = -\frac{(20-t)^{1.5}}{1.5\times 20^{1.5}}\Bigg|_0^1 = \frac{1}{1.5}\left(1-\left(\frac{19}{20}\right)^{1.5}\right) = 0.04937$$

$$q_x^{(2)} = \Pr(T \le 1\ ,\ J = 2) = \int_0^1 f_{T,J}(t,2)\,dt$$

$$= \int_0^1 \frac{0.5(20-t)^{0.5}}{20^{1.5}}\,dt = -\frac{0.5(20-t)^{1.5}}{1.5\times 20^{1.5}}\Bigg|_0^1 = \frac{0.5}{1.5}\left(1-\left(\frac{19}{20}\right)^{1.5}\right) = 0.02468$$

$$q_x^{(1)} + q_x^{(2)} = 0.04937 + 0.02468 = 0.07405$$

$$q_x'^{(1)} + q_x'^{(2)} - q_x'^{(1)}\,q_x'^{(2)} = 0.05000 + 0.02532 - 0.05000\times 0.02532 = 0.07405$$

Solution 7.10

$$\mathring{e}_x^{(\tau)} = \int_0^\infty {}_tp_x^{(\tau)}\,dt = \int_0^{20}\left(\frac{20-t}{20}\right)^{1.5} dt = -\frac{(20-t)^{2.5}}{2.5\times 20^{1.5}}\Bigg|_0^{20} = -0 + \frac{20}{2.5} = 8$$

Solution 7.11

$$\Pr(J=1) = \int_0^{20} f_{T,J}(t,1)\,dt = \int_0^{20} \frac{(20-t)^{0.5}}{20^{1.5}}\,dt = -\frac{(20-t)^{1.5}}{1.5\times 20^{1.5}}\Bigg|_0^{20} = -0 + \frac{1}{1.5} = \frac{2}{3}$$

$$\Pr(J=2) = \int_0^{20} f_{T,J}(t,2)\,dt = \int_0^{20} \frac{0.5(20-t)^{0.5}}{20^{1.5}}\,dt = -\frac{0.5(20-t)^{1.5}}{1.5\times 20^{1.5}}\Bigg|_0^{20} = -0 + \frac{0.5}{1.5} = \frac{1}{3}$$

We could have also done this without integration:

$$\Pr(J=1 \mid T=t) = \frac{\mu^{(1)}(x+t)}{\mu^{(\tau)}(x+t)} = \frac{1/(20-t)}{1.5/(20-t)} = \frac{2}{3} \quad \text{for } 0 \le t < 20$$

$$\Rightarrow T \text{ and } J \text{ are independent } \Rightarrow \Pr(J=1) = \Pr(J=1 \mid T=t) = \frac{2}{3}$$

Solution 7.12

From the table we have:

$$q_x^{(1)} = \frac{4}{100} = 0.04 \quad , \quad q_x^{(2)} = \frac{6}{100} = 0.06$$

The SUDD relations are:

$$0.04 = q_x^{(1)} = q_x'^{(1)}\left(1 - 0.5\,q_x'^{(2)}\right)$$
$$0.06 = q_x^{(2)} = q_x'^{(2)}\left(1 - 0.5\,q_x'^{(1)}\right)$$

Let's try the iterative approach:

$$q_x'^{(1)} = \frac{0.04}{1 - 0.5\,q_x'^{(2)}} \quad , \quad q_x'^{(2)} = \frac{0.06}{1 - 0.5\,q_x'^{(1)}}$$
$$q_x'^{(2)} = 0.06 \;\Rightarrow\; q_x'^{(1)} = 0.041237 \;\Rightarrow\; q_x'^{(2)} = 0.061263 \;\Rightarrow\; q_x'^{(1)} = 0.041264$$
$$\Rightarrow\; q_x'^{(2)} = 0.061264 \;\Rightarrow\; q_x'^{(1)} = 0.041264 \;\Rightarrow\; q_x'^{(2)} = 0.061264$$

The 6-place stable results are: $q_x'^{(1)} = 0.041264$, $q_x'^{(2)} = 0.061264$

Solution 7.13

From the table we have:

$$q_x^{(1)} = \frac{4}{100} = 0.04 \quad , \quad q_x^{(1)} = \frac{6}{100} = 0.06 \quad , \quad q_x^{(\tau)} = 0.10 \quad , \quad p_x^{(\tau)} = 0.90$$

The MUDD model results in:

$$q_x'^{(1)} = 1 - p_x'^{(1)} = 1 - \left(p_x^{(\tau)}\right)^{q_x^{(1)}/q_x^{(\tau)}} = 1 - (0.90)^{0.04/0.10} = 0.041268$$
$$q_x'^{(2)} = 1 - p_x'^{(2)} = 1 - \left(p_x^{(\tau)}\right)^{q_x^{(2)}/q_x^{(\tau)}} = 1 - (0.90)^{0.06/0.10} = 0.061260$$

Solution 7.14

$$\text{SUDD} \;\Rightarrow\; {}_tp_x^{(\tau)} = {}_tp_x'^{(1)}\;{}_tp_x'^{(2)} = (1 - 0.041264t)(1 - 0.061264t)$$
$$= 1 - 0.102528\,t + 0.002528\,t^2$$
$$\text{MUDD} \;\Rightarrow\; {}_tp_x^{(\tau)} = 1 - t\,q_x^{(\tau)} = 1 - 0.10\,t$$

Solution 7.15

Under the MUDD model we saw that:

$$f_{T,J}(t,j) = {}_{k|}q_x^{(j)} \quad \text{where} \quad k=[t]$$

$$\Rightarrow \; f_{T,J}(0.5,1) = {}_{0|}q_x^{(1)} = \frac{d_x^{(1)}}{l_x^{(\tau)}} = \frac{4}{100}$$

Solution 7.16

We have the following generally valid relation:

$$0.04 = q_x^{(1)} = \int {}_tp_x'^{(2)} \, {}_tp_x'^{(1)} \, \mu^{(1)}(x+t)\,dt$$

Since decrement (1) follows the SUDD model, we have:

$${}_tq_x'^{(1)} = t\,q_x'^{(1)} \quad \Rightarrow \quad {}_tp_x'^{(1)} \, \mu^{(1)}(x+t) = q_x'^{(1)} \quad \text{for } 0 \le t \le 1$$

For decrement (2) we have:

$${}_tp_x'^{(2)} = \left(p_x'^{(2)}\right)^t$$

Substituting these results into the first equation results in the relation:

$$0.04 = q_x^{(1)} = \int_0^1 {}_tp_x'^{(2)} \, {}_tp_x'^{(1)} \, \mu^{(1)}(x+t)\,dt = \int_0^1 q_x'^{(1)} \left(p_x'^{(2)}\right)^t dt$$

$$= q_x'^{(1)} \left(-\frac{1-p_x'^{(2)}}{\ln\left(p_x'^{(2)}\right)} \right) = -\frac{q_x'^{(1)} \, q_x'^{(2)}}{\ln\left(1-q_x'^{(2)}\right)}$$

Solution 7.17

By the discrete time method, we have:

$$0.06 = q_x^{(2)} = \sum_{\substack{0 \le t_k \le 1 \\ \Pr(T_2 = t_k) \ne 0}} \left(\prod_{j \ne 2} {}_{t_k}p_x'^{(j)} \right) \Pr(T_2 = t_k) = \left(1 - 0.5q_x'^{(1)}\right) \Pr(T_2 = 0.5) = \left(1 - 0.5q_x'^{(1)}\right) q_x'^{(2)}$$

We also have the generally valid relation:

$$0.10 = q_x^{(1)} + q_x^{(2)} = q_x'^{(1)} + q_x'^{(2)} - q_x'^{(1)} q_x'^{(2)}$$

If you subtract the first relation from the generally valid relation, then you will have 2 relations that are identical to the relations that result if both decrements follow the SUDD model. So the result will be the same as in Question 7.12.

Solution 7.18

The random present value variable for the benefit is:

$$Z = \begin{cases} 1{,}000e^{-0.05T} & J=1 \\ 2{,}000e^{-0.05T} & J=2 \end{cases}$$

From the force formulas we can calculate the joint pdf $f_{T,J}(t,j)$:

$$\mu_{30}^{(1)}(t) = \frac{1}{60-t} \quad \text{for } 0 \le t < 60 \;\Rightarrow\; {}_tp_x'^{(1)} = \frac{60-t}{60} \quad \text{for } 0 \le t < 60 \;\; \text{(zero otherwise)}$$
$$\mu_{30}^{(2)}(t) = 0.01 \;\; \text{for } t > 0 \;\Rightarrow\; {}_tp_x'^{(2)} = e^{-0.01t}$$

$$f_{T,J}(t,1) = {}_tp_x^{(\tau)}\mu^{(1)}(x+t) = \left(\frac{60-t}{60}\right)e^{-0.01t} \times \frac{1}{60-t} = \frac{e^{-0.01t}}{60} \quad \text{for } 0 \le t < 60$$
$$f_{T,J}(t,2) = {}_tp_x^{(\tau)}\mu^{(2)}(x+t) = \left(\frac{60-t}{60}\right)e^{-0.01t} \times 0.01 = \frac{(60-t)e^{-0.01t}}{6{,}000} \quad \text{for } 0 \le t < 60$$

So the single benefit premium is:

$$\begin{aligned} E[Z] &= \sum_{j=1}^{2}\left(\int_0^\infty z_{t,j}\, f_{T,J}(t,j)\,dt\right) \\ &= \int_0^\infty 1{,}000e^{-0.05t} f_{T,J}(t,1)\,dt + \int_0^\infty 2{,}000e^{-0.05t} f_{T,J}(t,2)\,dt \\ &= \int_0^{60}\left(1{,}000\,e^{-0.05t}\right)\left(\frac{e^{-0.01t}}{60}\right)dt + \int_0^{60}\left(2{,}000\,e^{-0.05t}\right)\left(\frac{(60-t)e^{-0.01t}}{6{,}000}\right)dt \end{aligned}$$

Solution 7.19

$$\begin{aligned} \text{APV} &= \frac{1{,}000\,q_x^{(1)}}{1.05} + \frac{1{,}000\;{}_{1|}q_x^{(1)}}{1.05^2} + \frac{2{,}000\,q_x^{(2)}}{1.05} + \frac{2{,}000\;{}_{1|}q_x^{(2)}}{1.05^2} \\ &= \frac{1{,}000(0.035)}{1.05} + \frac{1{,}000(0.045)}{1.05^2} + \frac{2{,}000(0.001)}{1.05} + \frac{2{,}000(0.002)}{1.05^2} \\ &= 79.68254 \end{aligned}$$

Solution 7.20

Since the premiums are paid at time 0 and at time 1 (if surviving), the APV of premium is equal to:

$$P + P\,v\,p_x^{(\tau)} = P\left(1 + \frac{964/1,000}{1.05}\right) = 1.91810P$$

As a result of the equivalence principle, we have:

$$1.91810P = \text{APV of Premium} = \text{APV of Benefit} = 79.68254$$
$$\Rightarrow\ P = 41.54254$$