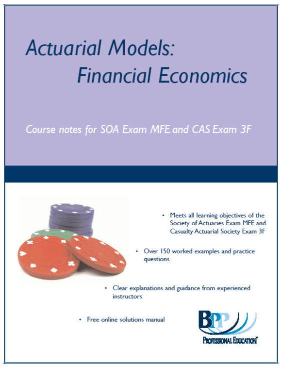

Financial economics (MFE)

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Solutions to practice questions – Chapter 2

Solution 2.1

(a) We need to equate the two possible values of the portfolio at time 1 to the derivative payoffs:

Upper node: $60\Delta + e^{0.05}B = 10$

Lower node: $30\Delta + e^{0.05}B = 25$

We can subtract these equations to find Δ:

$$60\Delta - 30\Delta = 10 - 25 \qquad \Rightarrow 30\Delta = -15 \qquad \Rightarrow \Delta = -0.5$$

We can then substitute this value of Δ into the equation for the upper node to find B:

$$60(-0.5) + e^{0.05}B = 10 \qquad B = e^{-0.05}\left[10 + 60(0.5)\right] = 40e^{-0.05} = 38.05$$

So a portfolio consisting of a short holding of 0.5 shares and a long position of 38.05 in cash would replicate the derivative.

(b) If the derivative is fairly priced, its value will be the same as the value of the replicating portfolio.

So: $\quad \textit{Value of derivative} = \Delta S + B = -0.5(40) + 38.05 = 18.05$

Solution 2.2

(a) When the risk-neutral probabilities are used, the expected rate of return earned by the underlying asset equals the risk-free interest rate earned by cash. It follows that the expected rate of return earned by any *derivative* based on the underlying asset (whose payoff can be replicated) is also equal to the risk-free rate.

(b) They are called "risk-neutral" probabilities because, if these probabilities applied, all the securities involved in the model would behave in the same way as a risk-free investment, in the sense that they would all earn the same rate on average as a holding in cash. So investors would not demand any extra return as a reward for taking on risk, *ie* they would be "risk-neutral".

Solution 2.3

We can find the risk-neutral probabilities, p^* and $1-p^*$, from first principles, since we know that the expected rate of return of the underlying asset must equal the risk-free interest rate:

$$110p^* + 95(1-p^*) = 100e^{0.25(0.04)}$$

So: $15p^* = 100e^{0.01} - 95 \quad \Rightarrow p^* = 0.4003$

So the risk-neutral probabilities are 0.4003 ("up") and 0.5997 ("down").

Alternatively, you can find p^* from the formula $p^* = \dfrac{e^{(r-\delta)h} - d}{u-d}$, with $r = 0.04$, $\delta = 0$, $h = 0.25$, $d = 0.95$ and $u = 1.10$.

(b) Since it is an at-the-money option, the strike price is $K = 100$. So the payoff function is $\max[100 - S_1, 0]$, *ie* 0 at the upper node and 5 at the lower node.

(c) The price of the option equals the expected payoff (calculated using the risk-neutral probabilities), discounted at the risk-free interest rate:

$$\textit{Value of put option} = e^{-0.01}\left[0.4003 \times 0 + 0.5997 \times 5\right] = 2.97$$

Solution 2.4

The key feature of an American option is that it allows the holder to exercise the option early if desired. Since the one-step binomial model only considers the initial position and the position at expiration, it provides no scope for considering the possibility of exercising at some intermediate time during the life of the option.

In the next chapter we will look at multi-step binomial trees, which get round this problem.

Solution 2.5

(a) $d = e^{(r-\delta)h - \sigma\sqrt{h}}$, $p^* = \dfrac{e^{(r-\delta)h} - d}{u-d}$ and $C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right]$.

(b) The completed table looks like this:

Volatility σ	"Up" ratio u	"Down" ratio d	Risk-neutral probabilities p^*, $1-p^*$	Option payoffs C_u, C_d	Option price C
0.2	1.1107	0.9094	0.4750, 0.5250	1.1071, 0	0.52
0.1	1.0565	0.9560	0.4875, 0.5125	0.5654, 0	0.27
0	1.0050	1.0050	0.5, 0.5	0.0501, 0.0501	0.05

When $\sigma = 0$, the value of p^* is not well-defined (because $u = d$ and the formula involves dividing by zero). We have used 0.5, which is the limiting value as σ becomes very small. (Try $\sigma = 0.000001$ on your calculator.)

(c) If we assume a higher volatility, we get a higher option price (*eg* 0.52 versus 0.27).

Even if we are unsure of the volatility, we can conclude from this model that the option is worth at least 0.05 (corresponding to zero volatility).

The values of the risk-neutral probabilities are such that $p^* \le \frac{1}{2} \le 1 - p^*$.

Solution 2.6

(a) This formula gives the value of a derivative calculated according to a one-step binomial model.

C is the fair value of the derivative, *ie* assuming there are no arbitrage opportunities in the market.

h is the length of the time-step in the model, which corresponds to the remaining life of the derivative.

r is the continuously-compounded risk-free interest rate, *ie* the rate earned on cash.

p^* and $1-p^*$ are the risk-neutral probabilities corresponding to the "up" and "down" branches.

C_u and C_d are the derivative payoffs corresponding to the "up" and "down" nodes.

(b) The one-step binomial model makes the following assumptions.

1. The derivative has only two possible payoffs.
2. These payoffs will be made at the fixed time h.
3. The derivative payoff depends only on the value of the underlying asset at the node reached.
4. The risk-free interest rate over the period has a known value and is the same for long and short cash positions.
5. It is possible to hold a short position in the underlying asset.
6. The market is arbitrage-free.

Solution 2.7

(a) The dividend yield δ must be replaced in the formulas with r_f, the interest rate on the foreign currency.

(b) The interest rate r and the dividend yield δ must be replaced in the formulas with 0.

Solution 2.8

At the upper node, each share will be worth 60 and each option will be worth $\max[50-60,0]=0$. So the total portfolio value will be $100\times 60+300\times 0=6000$.

At the lower node, each share will be worth 45 and each option will be worth $\max[50-45,0]=5$. So the total portfolio value will be $100\times 45+300\times 5=6000$.

So the portfolio has the same value either way. This means that (based on this model) we have a *risk-free portfolio*. Its value is not affected by the behavior of the underlying asset price.

In fact, if we knew the risk-free rate and the initial value of the underlying asset, we could use this portfolio to calculate the price of the put option. Since the portfolio will definitely be worth 6000 at expiration, its initial value must equal $6000e^{-rh}$, *ie* 6000 discounted at the risk-free rate. But its initial value is $100S_0+300P$, where S_0 and P are the initial values of the underlying asset and the put option. This tells us that $100S_0+300P=6000e^{-rh}$, which we can rearrange to get the formula $P=20e^{-rh}-\frac{1}{3}S_0$.

Solution 2.9

We saw in Question 2.1 that this option could be replicated using a replicating portfolio consisting of a short position in 0.5 shares and a long position of 38.05 in cash. According to our model, this portfolio replicates the option payoff and costs 18.05 to set up.

The discrepancy of $18.05 - 17.75 = 0.30$ is available to be taken as an arbitrage profit. To do this we need to buy one option (because it is available for less than its theoretical price), which will cost 17.75, and at the same time set up the "negative" of the replicating portfolio, which will bring in 18.05. We keep the remaining 0.30 as our arbitrage profit. The "negative" of the replicating portfolio involves buying 0.5 shares and borrowing 38.05 in cash (which will create a short position in cash).

At the expiration date, the value of the replicating portfolio should exactly match any payment we will receive from the derivative. So there will be no further cash flow at this time.