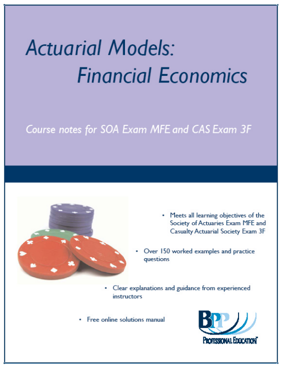

Financial economics (MFE)

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Solutions to practice questions – Chapter 6

Solution 6.1

Here we have:

$$A(T)=\frac{1}{6}(92+89+\cdots+104)=101$$

and $$G(T)=(92\times 89\times\cdots\times 104)^{1/6}=100.643$$

So the payoffs for the options are:

(a) $\max\left[A(T)-K,0\right]=\max[101-100,0]=1$

(b) $\max\left[G(T)-S_T,0\right]=\max[100.643-104,0]=0$

Solution 6.2

(a) Since $G(T)\le A(T)$ (from the arithmetic-geometric mean inequality), the payoff from the geometric average version, which is $\max\left[G(T)-K,0\right]$, will be slightly less than the original payoff, which was $\max\left[A(T)-K,0\right]$ (or both will be zero). So the price of the geometric average version will be slightly lower.

(b) Changing to monthly averaging will reduce the volatility of $A(T)$. An option with a less volatile payoff function will have a lower value (because the potential for really big payoffs is reduced). So the price will be lower.

(c) The payoff function would change from $\max\left[A(T)-100,0\right]$ to $\max\left[S_T-A(T),0\right]$. There is no simple relationship between the prices of these two options. So the price could go up or down.

(d) The price of the American version cannot be lower than the price of the European version. If the underlying share (stock) pays dividends, the American version will have a higher price, but if it does not, the two prices will be the same (for the same reasons as for vanilla call options on non-dividend-paying stocks).

Solution 6.3

Average *price* Asian options (either calls or puts) based on a *geometric* average are the easiest to value using the Black-Scholes method.

Solution 6.4

Put-call parity doesn't apply to *American* options of any sort.

The payoffs from *European* average price options based on an arithmetic average (say) are:

$$Payoff\ (call) = \max\left[A(T) - K, 0\right] \quad \text{and} \quad Payoff\ (put) = \max\left[K - A(T), 0\right]$$

Let's follow the method used to derive put-call parity in Chapter 1.

A portfolio consisting of a long position in 1 call option and a short position in 1 put option will have a payoff of:

$$Payoff\ (portfolio) = \max\left[A(T) - K, 0\right] - \max\left[K - A(T), 0\right] = A(T) - K$$

To establish a put-call parity relationship, we need to set up another portfolio now that will accumulate to have the same value as this. The K element is easy (we can borrow Ke^{-rT} in cash), but there is no portfolio whose value will develop naturally in line with $A(T)$, or $G(T)$ either.

So put-call parity does not apply to either European or American Asian options.

Solution 6.5

One example would be a corporation that makes sales overseas and receives payments each month that are then converted to dollars. If the corporation wants to reduce its exposure to currency risk, it could use a currency option. However, a (European) vanilla option would not be very effective since the payoff would depend only on the exchange rate on the expiration date, which may be very different from the previous monthly exchange rates. An Asian option would be better, since the payoff would reflect the average exchange rate over the whole period.

Solution 6.6

Mathematically, the combined payoff from the down-and-out put option and the short put is:

$$\begin{aligned} Payoff &= \max(K - S_T, 0) \times I\left(\min_{0 \le t \le T} S_t > H\right) - \max(K - S_T, 0) \\ &= -\max(K - S_T, 0) \times \left\{1 - I\left(\min_{0 \le t \le T} S_t > H\right)\right\} \\ &= -\max(K - S_T, 0) \times I\left(\min_{0 \le t \le T} S_t \le H\right) \end{aligned}$$

This matches the payoff from a short position in a down-and-in put option.

You can reach the same conclusion without using algebra by considering the two possibilities:

1. If the share price doesn't fall below H, the payoff from the barrier option will exactly cancel out with the vanilla put option.
2. If the share price does fall below H, there will just be the payoff from the short put.

So the payoff from the combination is the same as for a short put, but only if the share price falls below H – in other words a short down-and-in put.

Solution 6.7

The option will only get knocked in if it reaches the 113.74 node. There will then be two possible payoffs: $120.77-100=20.77$ or $109.28-100=9.28$.

now	*3 months*	*6 months*	*9 months*	*1 year*
				120.77
			113.74	
		107.11		**109.28**
	100.87		102.91	
95		96.92		98.88
	91.27		93.12	
		87.70		89.47
			84.26	
				80.95

To reach the 120.77 node, we have to follow the path "up-up-up-up", which has a risk-neutral probability of $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{16}$.

To reach the 109.28 node, passing through the 113.74 node (so that the barrier option gets knocked in), we have to follow the path "up-up-up-down", which also has a risk-neutral probability of $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{16}$.

The value of the barrier option is then calculated as the discounted risk-neutral expectation of the payoff:

$$1.01^{-4}\times\left(\tfrac{1}{16}\times 20.77+\tfrac{1}{16}\times 9.28\right)=1.80$$

This value is probably not very reliable because the calculations we carried out depended only on the underlying asset prices in one small corner of the tree where the option was knocked in.

Solution 6.8

The put-call parity relationship for compound options tells us that:

$$\begin{bmatrix}\textit{Value of}\\ \textit{Compound Call Option}\end{bmatrix}-\begin{bmatrix}\textit{Value of}\\ \textit{Compound Put Option}\end{bmatrix}=\begin{bmatrix}\textit{Value of}\\ \textit{Underlying Option}\end{bmatrix}-xe^{-rt_1}$$

Substituting the parameter values given:

$$9-3=25-xe^{-0.05}$$

So: $x=(25-9+3)e^{0.05}=19.97$

So the strike price for the compound options appears to be 20.

The strike prices for options are usually chosen to be "round" numbers.

Solution 6.9

We saw in Chapter 1 that the price of a vanilla call option can never exceed the price of the underlying asset (and they can only be equal if $K=0$). Therefore (b) has a lower price than (a).

If we now let S_t^* denote the price of the vanilla call option in (b), we can use exactly the same logic to deduce that (c) has a lower price than (b).

So, combining these results, we conclude that:

$$Price(c) \le Price(b) \le Price(a)$$

Solution 6.10

(a) Your payoff function in respect of each share is:

$$Payoff = S_1 \times I(S_1 < 5)$$

The payoff function from a long position in a gap put option, with $K_1 = 0$, $K_2 = 5$ and $T = 1$, is:

$$\begin{aligned} Payoff\,(gap\ put) &= (K_1 - S_T) \times I(S_T < K_2) \\ &= (0 - S_1) \times I(S_1 < 5) \\ &= -S_1 \times I(S_1 < 5) \end{aligned}$$

Comparing these, we see that your payoff is the same as a *short* position in this option.

(b) If we use the Black-Scholes model, the value of your options (per share) is therefore:

$$\begin{aligned} -P_{Eur}^{gap} &= -\left[K_1 e^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)\right] \\ &= -\left[0e^{-r} N(-d_2) - Se^{-\delta} N(-d_1)\right] \\ &= Se^{-\delta} N(-d_1) \end{aligned}$$

Using the parameter values given ($S=10$, $\sigma=0.3$, $\delta=0$ and $r=0.05$), we get:

$$d_1 = \frac{\ln\left(\dfrac{Se^{-\delta T}}{K_2 e^{-rT}}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\dfrac{10}{5e^{-0.05}}\right) + \frac{1}{2}(0.3)^2(1)}{0.3\sqrt{1}} = 2.6272$$

So: $-P_{Eur}^{gap} = 10N(-2.63) = 10 \times 0.0043 = 0.0430$

So the value of his gift of options on 100 shares is \$4.30. (His shares are currently worth \$1000.)

Solution 6.11

The payoffs from gap options are:

Payoff for a gap *call* option $= (S_T - K_1) \times I(S_T > K_2)$

Payoff for a gap *put* option $= (K_1 - S_T) \times I(S_T < K_2)$

For the gap *call* option, the payoff function will be negative if the amount $(S_T - K_1)$ is negative, *ie* if $S_T < K_1$, and the indicator function has a value of 1, *ie* if $S_T > K_2$. So, to get a negative payoff, we would need $K_2 < S_T < K_1$.

For the gap *put* option, the payoff function will be negative if the amount $(K_1 - S_T)$ is negative, *ie* if $S_T > K_1$, and the indicator function has a value of 1, *ie* if $S_T < K_2$. So, to get a negative payoff, we would need $K_1 < S_T < K_2$.

Solution 6.12

The Black-Scholes formula for the price of an option to exchange 1 unit of an underlying asset (with price S and "dividend yield" δ_S) for 1 unit of a strike asset (with price K and "dividend yield" δ_K) at time T is:

$$C_{Eur}^{exchange} = Se^{-\delta_S T} N(d_1) - Ke^{-\delta_K T} N(d_2)$$

where $$d_1 = \frac{\ln\left(\dfrac{Se^{-\delta_S T}}{Ke^{-\delta_K T}}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

and $$\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K}.$$

To apply this in (a), from the point of view of a trader in New York, the parameter values are:

$S = 2$ (because initially £1 is worth \$2)

$\delta_S = 0.06$ (because the underlying pound is earning 6% interest)

$K = 2$ (because the agreed strike price for the pound is \$2)

$\delta_K = 0.04$ (because the US dollar is earning 4% interest)

$T = 0.5$ (= 6 months)

$\sigma = 0.2$ (given)

So we get:

$$d_1 = \frac{\ln\left(\dfrac{2e^{-0.06(0.5)}}{2e^{-0.04(0.5)}}\right) + \frac{1}{2}(0.2)^2(0.5)}{0.2\sqrt{0.5}} = 0$$

$$d_2 = 0 - 0.2\sqrt{0.5} = -0.1414$$

and $$\begin{aligned} C_{Eur}^{exchange} &= 2e^{-0.03}N(0) - 2e^{-0.02}N(-0.14) \\ &= 2e^{-0.03} \times 0.5 - 2e^{-0.02} \times 0.4443 \\ &= \$0.10 \end{aligned}$$

For (b), from the point of view of a trader in London, the parameter values are:

$S = 1$ (because initially \$2 is worth £1)

$\delta_S = 0.04$ (because the underlying dollar is earning 4% interest)

$K = 1$ (because the agreed strike price for the \$2 is £1)

$\delta_K = 0.06$ (because the British pound is earning 6% interest)

$T = 0.5$ (= 6 months)

$\sigma = 0.2$ (given)

So we get:

$$d_1 = \frac{\ln\left(\frac{1e^{-0.04(0.5)}}{1e^{-0.06(0.5)}}\right) + \frac{1}{2}(0.2)^2(0.5)}{0.2\sqrt{0.5}} = 0.1414$$

$$d_2 = 0.1414 - 0.2\sqrt{0.5} = 0$$

and

$$\begin{aligned} C_{Eur}^{exchange} &= 1e^{-0.02}N(0.14) - 1e^{-0.03}N(0) \\ &= 1e^{-0.02} \times 0.5557 - 1e^{-0.03} \times 0.5 \\ &= £0.06 \end{aligned}$$

At the current exchange rate, the price calculated in (b) is equivalent to $2 \times 0.06 = \$0.12$. So the prices do *not* correspond.

The reason for this is that the New York trader will exercise if the \$ is weak (relative to the £), whereas the London trader will exercise if the \$ is strong. So the optionality present in the two options is not the same.

This is different from the duality situation we illustrated in Example 1.6 in Chapter 1. In that example, both parties will exercise if the dollar is weak and the optionality was the same.

