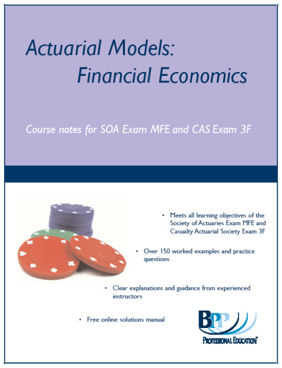

Financial economics (MFE)

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Solutions to practice questions – Chapter 8

Solution 8.1

(a) The Vasicek model and the Cox-Ingersoll-Ross model incorporate mean reversion.

(b) The Rendelman-Bartter and the Cox-Ingersoll-Ross model prevent negative interest rates arising.

(c) All the models described are diffusion models based on Brownian motion. So none of them allow discontinuous jumps to occur.

(d) The Black model can be used to value options based on bonds or interest rates

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Solution 8.2

We can find the SDE for the bond price by applying Itô's lemma to the original process $r(t)$ and the time t:

$$dP = \frac{\partial P}{\partial r}dr + \tfrac{1}{2}\frac{\partial^2 P}{\partial r^2}(dr)^2 + \frac{\partial P}{\partial t}dt$$

If we now use the SDE for $r(t)$ and simplify, we get:

$$\begin{aligned} dP &= \frac{\partial P}{\partial r}[0.1(0.05-r)dt + 0.01dZ] + \tfrac{1}{2}\frac{\partial^2 P}{\partial r^2}[0.1(0.05-r)dt + 0.01dZ]^2 + \frac{\partial P}{\partial t}dt \\ &= \frac{\partial P}{\partial r}[0.1(0.05-r)dt + 0.01dZ] + \tfrac{1}{2}\frac{\partial^2 P}{\partial r^2}(0.01)^2 dt + \frac{\partial P}{\partial t}dt \\ &= \left\{0.1(0.05-r)\frac{\partial P}{\partial r} + 0.00005\frac{\partial^2 P}{\partial r^2} + \frac{\partial P}{\partial t}\right\}dt + 0.01\frac{\partial P}{\partial r}dZ \end{aligned}$$

Since $E^*[dZ] = 0$, it follows that:

$$E^*[dP] = \left\{0.1(0.05-r)\frac{\partial P}{\partial r} + 0.00005\frac{\partial^2 P}{\partial r^2} + \frac{\partial P}{\partial t}\right\}dt$$

If the calculations are based on risk-neutral probabilities, the bond price will be increasing in value at the risk-free rate on average, so that:

$$E^*[dP] = rPdt$$

Comparing these two equations, we can conclude that:

$$0.1(0.05-r)\frac{\partial P}{\partial r}+0.00005\frac{\partial^2 P}{\partial r^2}+\frac{\partial P}{\partial t}=rP$$

This gives us a partial differential equation satisfied by the bond price P.

Solution 8.3

We need to calculate $B(0,10)$ which is:

$$B(0,10)=\frac{1-e^{-0.2(10)}}{0.2}=4.3233\,.$$

The price of the 10-year bond is then:

$$P[0.02,0,10]=A(0,10)e^{-B(0,10)\times 0.02}=0.7565e^{-4.3233\times 0.02}=0.6938 \quad \text{(or } \$69.38)$$

The 10-year spot rate of interest is the constant interest rate over the next 10 years implied by the current price of a zero-coupon bond maturing at time 10. It can be calculated as:

$$s(0,10)=-\frac{1}{10}\ln 0.6938=0.0366$$

Alternatively, you can work from the equation of value:

$$100e^{-10s(0,10)}=69.38$$

So the 10-year spot rate based on this model is 3.66%.

Solution 8.4

Using the equation $P[r(t),t,T]=A(t,T)e^{-B(t,T)r(t)}$ and the time-homogeneous property, which tells us that $A(2,7)=A(0,5)$ and $B(2,7)=B(0,5)$, we can calculate the value of this bond as:

Vasicek: $P[0.04,2,7]=A(2,7)e^{-B(2,7)\times 0.04}=0.9131e^{-3.1606\times 0.04}=0.8047$

CIR: $P[0.04,2,7]=A(2,7)e^{-B(2,7)\times 0.04}=0.9127e^{-3.1210\times 0.04}=0.8056$

So the price is \$80.47 under the Vasicek model and \$80.56 under the CIR model.

Solution 8.5

We can use Black's model. The value of a put option is:

$$P(0,T)[KN(-d_2)-FN(-d_1)]$$

where $d_1=\dfrac{\ln(F/K)+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$ and $d_2=d_1-\sigma\sqrt{T}$.

Here: $F=96$, $K=90$, $\sigma=0.2$, $T=1$, $P(0,T)=0.95$

So: $d_1 = \dfrac{\ln(96/90) + \frac{1}{2}(0.2)^2(1)}{0.2\sqrt{1}} = 0.4227$ and $d_2 = 0.4227 - 0.2\sqrt{1} = 0.2227$

$$\begin{aligned} \textit{Value of put option} &= 0.95\left[90 \times N(-0.22) - 96 \times N(-0.42)\right] \\ &= 0.95\left[90 \times 0.4129 - 96 \times 0.3372\right] \\ &= 4.55 \end{aligned}$$

Solution 8.6

This bond will pay \$5 at time 1 and \$105 at time 2. So its value is:

$$e^{-0.05} \times \$5 + e^{-0.05} \times \left(0.6 \times e^{-0.06} \times \$105 + 0.4 \times e^{-0.04} \times \$105\right) = \$99.58$$

Solution 8.7

The interest rate for the first year is 5.00%.

So the first caplet will pay:

$$\$1{,}000 \times \max[0.05 - 0.04, 0] = \$10.00$$

and its value at time 0 is:

$$\frac{\$10.00}{1.05} = \$9.52$$

The interest rate for the second year will be either 7.72% or 6.32%.

So the second caplet will pay either:

$$\$1{,}000 \times \max[0.0772 - 0.04, 0] = \$37.20$$

or $\$1{,}000 \times \max[0.0632 - 0.04, 0] = \$23.20$

For the BDT model, the risk-neutral probabilities are all 0.5. So the value of the second caplet is:

$$\frac{1}{1.05} \times \left(0.5 \times \frac{\$37.20}{1.0772} + 0.5 \times \frac{\$23.20}{1.0632}\right) = \$26.84$$

So the value of the entire cap is:

$$\$9.52 + \$26.84 = \$36.36$$

Solution 8.8

The yield on the 1-year bond is equal to R_0. So $R_0 = 6\%$.

The forward volatility for the two-year bond is equal to σ_1. So $\sigma_1 = 10\%$.

The price of the 2-year bond is:

$$\frac{1}{1+R_0} \times \left(0.5 \times \frac{1}{1+R_1 e^{2\sigma_1}} + 0.5 \times \frac{1}{1+R_1}\right) = \frac{1}{1+0.06} \times \left(0.5 \times \frac{1}{1+R_1 e^{2(0.1)}} + 0.5 \times \frac{1}{1+R_1}\right)$$

But we know (from the yield given) that this bond price is $\frac{1}{1.05^2}$.

So: $$\frac{1}{1+0.06}\times\left(0.5\times\frac{1}{1+R_1e^{2(0.1)}}+0.5\times\frac{1}{1+R_1}\right)=\frac{1}{1.05^2}$$

$$\frac{0.5}{1.06}\times\left(\frac{1}{1+R_1e^{0.2}}+\frac{1}{1+R_1}\right)=\frac{1}{1.05^2}$$

$$\frac{1}{1+R_1e^{0.2}}+\frac{1}{1+R_1}=\frac{1}{1.05^2}\times\frac{1.06}{0.5}=1.9229$$

Clearing the fractions:

$$(1+R_1)+(1+R_1e^{0.2})=1.9229(1+R_1e^{0.2})(1+R_1)$$

$$2+R_1(1+e^{0.2})=1.9229\left[1+R_1(1+e^{0.2})+R_1^2e^{0.2}\right]$$

$$2+2.2214R_1=1.9229+4.2715R_1+2.3486R_1^2$$

$$\Rightarrow \quad 2.3486R_1^2+2.0501R_1-0.0771=0$$

Solving this equation using the quadratic formula, we find that the relevant root is $R_1=3.61\%$.

The other root is negative, so we can discard it.
