



Actuarial Models

Second Edition

By Michael A Gauger

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Solutions to practice questions – Chapter 1

Solution 1.1

$${}_2p_0 = \frac{l_2}{l_0} = \frac{985}{1,000}, \quad {}_2|q_0 = \frac{d_2}{l_0} = \frac{3}{1,000}, \quad {}_4|{}_2q_3 = \frac{l_7 - l_9}{l_3} = \frac{9}{982}$$

$$p_4 = \frac{l_5}{l_4} = \frac{976}{979}, \quad q_5 = \frac{d_5}{l_5} = \frac{4}{976}$$

Solution 1.2

The probability ${}_2|q_1 = 0.015$ is the probability that a life currently age 1 will die between ages 3 and 4. View each of the 20 lives age 1 in the group as a trial, where ‘success’ means that the life dies between ages 3 and 4. Then the random number of deaths follows a binomial distribution with $n = 20$ trials and $p = {}_2|q_1 = 0.015$. The expected number of deaths is $np = 0.30$ and the variance in the number of deaths is $npq = 0.29550$.

Solution 1.3

The probability function is: $\Pr(K = k) = d_k / l_0$. From the table we have:

$$l_0 = 100, \quad d_0 = l_0 - l_1 = 54, \quad d_1 = 27, \quad d_2 = 13, \quad d_3 = 6$$

So it follows that $\Pr(K = 0) = 54 / 100$, $\Pr(K = 1) = 27 / 100$, and so on.

Solution 1.4

If you are given a formula for the force function, then the survival function is the obvious choice for the first calculation:

$$\mu(x) = \frac{3}{2+x} \Rightarrow \int_0^x \mu(y) dy = (3 \ln(2+y)) \Big|_0^x = \left(\ln \left(\frac{2+x}{2} \right) \right)^3 \Rightarrow$$

$$s_X(x) = \exp \left(- \int_0^x \mu(y) dy \right) = \left(\frac{2}{2+x} \right)^3 \Rightarrow F_X(x) = 1 - s_X(x) = 1 - \left(\frac{2}{2+x} \right)^3 \Rightarrow$$

$$f_X(x) = F'_X(x) = \frac{2^3 \times 3}{(2+x)^4}$$

Solution 1.5

$$p_1 = \frac{l_2}{l_1} = \frac{s_X(2)}{s_X(1)} = \frac{(2/(2+2))^3}{(2/(2+1))^3} = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

$${}_{2|}q_1 = \frac{l_3 - l_4}{l_1} = \frac{s_X(3) - s_X(4)}{s_X(1)} = \frac{\left(\frac{2}{5}\right)^3 - \left(\frac{2}{6}\right)^3}{\left(\frac{2}{3}\right)^3} = \frac{91}{1,000}$$

Solution 1.6

$$\mu(x) = \frac{0.5}{100-x} \Rightarrow \int_0^x \mu(y) dy = 0.5 \left(-\ln(100-y) \Big|_0^x \right) = \ln \left(\left(\frac{100}{100-x} \right)^{0.5} \right) \Rightarrow$$

$$s_X(x) = \exp \left(-\ln \left(\left(\frac{100}{100-x} \right)^{0.5} \right) \right) = \left(\frac{100-x}{100} \right)^{0.5} \Rightarrow F_X(x) = 1 - \left(\frac{100-x}{100} \right)^{0.5} \Rightarrow$$

$$f_X(x) = \frac{1}{20(100-x)^{0.5}}$$

Solution 1.7

$$s_X(x) = \left(\frac{100-x}{100} \right)^{0.5} \Rightarrow {}_{20}p_{40} = \frac{s_X(60)}{s_X(40)} = \sqrt{\frac{40}{60}} = 0.81650$$

$${}_{20|}{}_{20}q_{40} = {}_{20}p_{40} \times (1 - {}_{20}p_{60}) = \sqrt{\frac{40}{60}} \times \left(1 - \sqrt{\frac{20}{40}} \right) = 0.23915$$

Solution 1.8

$$\mu(x) = \frac{1.1}{100+x} \Rightarrow \int_0^x \mu(y) dy = 1.1 \ln \left(\frac{100+x}{100} \right) \Rightarrow$$

$$s_X(x) = \exp \left(-1.1 \ln \left(\frac{100+x}{100} \right) \right) = \left(\frac{100}{100+x} \right)^{1.1} \Rightarrow$$

$${}_t p_{20} = \frac{s_X(20+t)}{s_X(20)} = \left(\frac{100}{100+20+t} \right)^{1.1} / \left(\frac{100}{100+20} \right)^{1.1} = \left(\frac{120}{120+t} \right)^{1.1}$$

Solution 1.9

For the curtate lifetime at age 20 to be less than 2, we must have $K(20) = 0$ or 1 . That means that (20) must die within 2 years. The probability is 1 minus the probability of surviving the next 2 years:

$${}_2q_{20} = \Pr(K(20) = 0 \text{ or } 1) = 1 - s_{T(20)}(2) = 1 - \left(\frac{120}{122}\right)^{1.1} = 0.01802$$

Solution 1.10

$$\mu(x) = 0.015 \Rightarrow s_X(x) = \exp\left(-\int_0^x 0.015 dy\right) = e^{-0.015x} \Rightarrow$$

$$f_{T(20)}(t) = {}_t p_{20} \mu(20+t) = \frac{s_X(20+t)}{s_X(20)} \times \mu(20+t) = \frac{0.015e^{-0.015(20+t)}}{e^{-0.015 \times 20}} = 0.015e^{-0.015t}$$

Solution 1.11

Start by computing a formula for the probability function:

$$\Pr(K(20) = k) = {}_k|q_{20} = {}_k p_{20} - {}_{k+1} p_{20} = e^{-0.015(k)} - e^{-0.015(k+1)} = (1 - e^{-0.015})(e^{-0.015k})$$

The probability that the curtate lifetime exceeds 1 is:

$$1 - \Pr(K(20) = 0) - \Pr(K(20) = 1) = 1 - (1 - e^{-0.015}) - (1 - e^{-0.015})e^{-0.015} = 0.97045$$

Note: You could have just as easily used $\Pr(K(20) \geq 2) = e^{-2 \times 0.015}$.

Solution 1.12

$$l_x = 1,000(100 - x)^{0.95} \Rightarrow T_x = \int_x^{100} l_y dy = 1,000 \left(-\frac{(100 - x)^{1.95}}{1.95} \right) \Bigg|_x^{100} = \frac{1,000(100 - x)^{1.95}}{1.95}$$

Solution 1.13

Use the formula developed in Solution 1.12 and the fact that $T_{20} - T_{25}$ is the number of people-years between ages 20 and 25 lived by the l_{20} lives that survive to age 20:

$${}^{\circ}e_{20:\overline{5}|} = \frac{T_{20} - T_{25}}{l_{20}} = \frac{\frac{1,000(80)^{1.95}}{1.95} - \frac{1,000(75)^{1.95}}{1.95}}{1,000(80)^{0.95}} = \frac{80^{1.95} - 75^{1.95}}{1.95 \times 80^{0.95}} = 4.85141$$

Solution 1.14

$$\mu(x)=0.015 \Rightarrow {}_t p_x = e^{-0.015t} \Rightarrow \overset{\circ}{e}_{x:\overline{1}|} = \int_0^1 {}_t p_x dt = \frac{1 - e^{-0.015}}{0.015} = 0.99254 \Rightarrow$$

$$a(x) = \frac{\overset{\circ}{e}_{x:\overline{1}|} - p_x}{q_x} = \frac{0.99254 - e^{-0.015}}{1 - e^{-0.015}} = 0.49875$$

Note: When the force of mortality is constant, on average, deaths during a given year of age occur slightly before mid-year.

Solution 1.15

$$q_{20} = 1 - p_{20} = 1 - e^{-0.015} = 0.01489$$

$$m_{20} = \frac{\int_0^1 {}_t p_{20} \mu(20+t) dt}{\int_0^1 {}_t p_{20} dt} = \frac{\int_0^1 {}_t p_{20} 0.015 dt}{\int_0^1 {}_t p_{20} dt} = 0.01500$$

Note. When the force of mortality is constant, the central rate at age x is the same as the force.

Solution 1.16

$$l_x = 1,000(100-x)^{0.75} \Rightarrow \mu(x) = \frac{-l'_x}{l_x} \Rightarrow l_x \mu(x) = -l'_x = 750 / (100-x)^{0.25}$$

Solution 1.17

$$q_{30} = 0.01, q_{31} = 0.02, q_{32} = 0.03 \text{ and the UDD} \Rightarrow$$

$$\mu(31.4) = \mu(31+0.4) = \frac{q_{31}}{1-0.4q_{31}} = \frac{0.02}{1-0.4 \times 0.02} = 0.02016$$

$${}_{1.4} p_{30} = p_{30} \times {}_{0.4} p_{31} = (1-0.01)(1-0.4 \times 0.02) = 0.98208$$

$$f_{T(30)}(1.4) = {}_{1.4} p_{30} \mu(31.4) = 0.01980$$

Solution 1.18

A person age 30 who survives 3 years gets credit for 3 years of life in both $e_{30:\overline{3}|}$ and $\overset{\circ}{e}_{30:\overline{3}|}$. A person who dies during the next 3 years gets more credit in the calculation of $\overset{\circ}{e}_{30:\overline{3}|}$ for part of the final year of life. Since $a(x)=0.5$ for all ages under the UDD assumption, it seems reasonable that $\overset{\circ}{e}_{30:\overline{3}|} = e_{30:\overline{3}|} + 0.5 {}_3 q_{30}$.

$$e_{30:\overline{3}|} = p_{30} + {}_2 p_{30} + {}_3 p_{30} = 0.99 + (0.99 \times 0.98) + (0.99 \times 0.98 \times 0.97) = 2.90129$$

$$\overset{\circ}{e}_{30:\overline{3}|} = e_{30:\overline{3}|} + 0.5 {}_3 q_{30} = 2.90129 + 0.5 \times (1 - 0.99 \times 0.98 \times 0.97) = 2.93075$$

Solution 1.19

$$\begin{aligned}
 l_x &= 1,000e^{-0.02x} \Rightarrow p_x = l_{x+1}/l_x = e^{-0.02} \text{ for all } x \\
 {}_3q_{[20]} &= 1 - {}_3p_{[20]} = 1 - p_{[20]} \times p_{[20]+1} \times p_{22} \\
 &= 1 - (1 - 0.8q_{20})(1 - 0.9q_{21})p_{22} \\
 &= 1 - (1 - 0.8(1 - p_{20}))(1 - 0.9(1 - p_{21}))p_{22} \\
 &= 1 - (1 - 0.8(1 - e^{-0.02}))(1 - 0.9(1 - e^{-0.02}))p_{22} \\
 &= 0.05252
 \end{aligned}$$

Solution 1.20

$$l_{[20]} = \frac{l_{22}}{{}_2p_{[20]}} = \frac{1,000e^{-0.02 \times 22}}{(1 - 0.8(1 - e^{-0.02}))(1 - 0.9(1 - e^{-0.02}))} = 666.28$$

Solution 1.21

Note first that we have:

$$\mu(60+t) = \frac{\text{density function}}{\text{survival function}} = \frac{-({}_t p_{60})'}{{}_t p_{60}}$$

For the UDD assumption:

$${}_t p_{60} = 1 - tq_{60} \Rightarrow \mu(60+t) = \frac{q_{60}}{1 - tq_{60}} \Rightarrow \mu(60.4) = \frac{q_{60}}{1 - 0.4q_{60}} = \frac{0.02}{1 - 0.4 \times 0.02} = 0.02016$$

For the exponential assumption:

$$\mu(60+t) = -\ln(p_{60}) \Rightarrow \mu(60.4) = -\ln(0.98) = 0.02020$$

For the hyperbolic assumption:

$$\begin{aligned}
 {}_t p_{60} &= \frac{p_x}{p_x + tq_x} \Rightarrow \mu(60+t) = \frac{-({}_t p_{60})'}{{}_t p_{60}} = \frac{p_{60} + tq_{60}}{p_{60}} \times \frac{p_{60} q_{60}}{(p_{60} + tq_{60})^2} = \frac{q_{60}}{p_{60} + tq_{60}} \\
 \Rightarrow \mu(60.4) &= \frac{q_{60}}{p_{60} + 0.4q_{60}} = 0.02024
 \end{aligned}$$