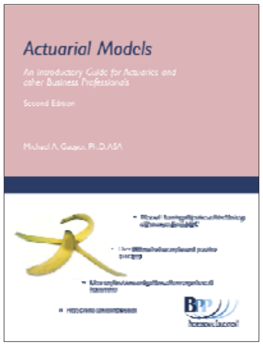

Actuarial Models

Second Edition

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Solutions to practice questions – Chapter 2

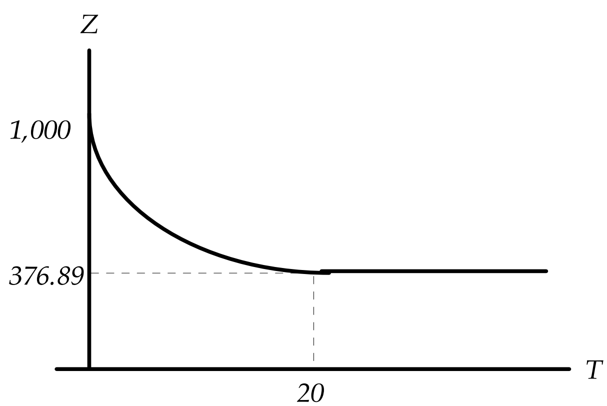
Solution 2.1

$$Z = \begin{cases} 1{,}000\ v^{T(45)} & \text{if } T(45) \le 20 \\ 1{,}000\ v^{20} & \text{if } T(45) > 20 \end{cases}$$

Solution 2.2

If death occurs at age 55.8, then $T(45) = 55.8 - 45 = 10.8$: $Z = 1{,}000 \times 1.05^{-10.8} = 590.41$.

If death occurs at age 70.2, then (45) was alive at age 65 and received the 1,000 at that time: $Z = 1{,}000 \times 1.05^{-20} = 376.89$.



Solution 2.3

The APV is denoted by $1{,}000\,\bar{A}_{45:\overline{20|}}$. The *A* indicates the actuarial present value of an insurance where the benefit amount is 1. (Assurance is an older term for insurance. It is where the A in APV symbols originates.) The over-bar indicates that death benefits are paid immediately on death. The symbol $45:\overline{20|}$ indicates that payment is made when a life age 45 dies, or when a 20-year period is over, whichever comes first. The coefficient 1,000 in front of the *A* indicates an increase from a face value of 1 to a face value of 1,000.

Solution 2.4

$$E[Z] = \underbrace{\int_0^{20} 1{,}000\ v^t\ f_{T(45)}(t)\,dt}_{\text{APV of death benefits}} + \underbrace{\int_{20}^{\omega-45} 1{,}000\ v^{20}\ f_{T(45)}(t)\,dt}_{\substack{\text{APV of the pure endowment} \\ =\ 1{,}000\ v^{20}\ {}_{20}p_{45}}}$$

Solution 2.5

From $\mu(x)=0.02$, it follows that:

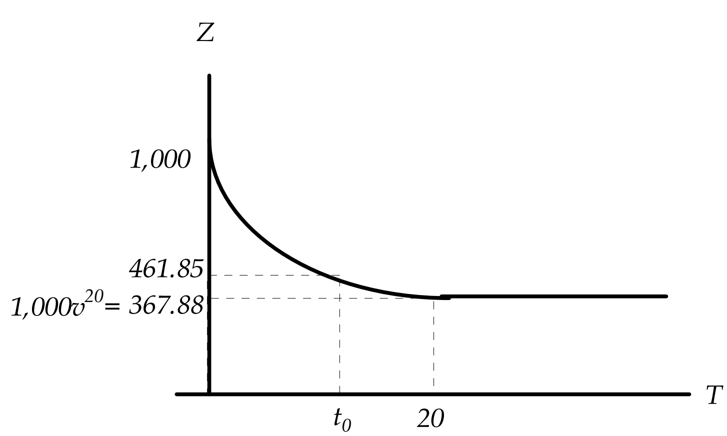
$${}_tp_{45} = e^{-0.02t} \quad , \quad f_{T(45)}(t) = 0.02\,e^{-0.02t} \ \text{ for } 0 < t < \infty$$

As a result, we have:

$$\begin{aligned} E[Z] &= \int_0^{20} 1{,}000\ v^t\ f_{T(45)}(t)\,dt + \int_{20}^{\omega-45} 1{,}000\ v^{20}\ f_{T(45)}(t)\,dt \\ &= 1{,}000\left(\int_0^{20} e^{-0.05t}\,0.02\,e^{-0.02t}\,dt + e^{-0.05\times 20}\,e^{-0.02\times 20}\right) \\ &= 1{,}000\left(\left(-\frac{0.02e^{-0.07t}}{0.07}\Bigg|_0^{20}\right) + e^{-1.4}\right) = 1{,}000\left(\frac{0.02\left(1-e^{-1.4}\right)}{0.07} + e^{-1.4}\right) = 461.85 \end{aligned}$$

Solution 2.6

You are asked to determine $\Pr\left(Z \le 1{,}000\,\bar{A}_{45:\overline{20}|}\right) = \Pr\left(Z \le 461.85\right)$. From the graph below we can see that this probability is the same as $\Pr\left(T(45) \ge t_0\right)$ where $1{,}000\,e^{-0.05t_0} = 461.85$: $t_0 = 15.45$



Now $\Pr\left(T(45) \ge t_0\right) = {}_{t_0}p_{45} = e^{-\mu t_0} = e^{-0.02t_0} = e^{-0.02\times 15.45} = 0.73418$

Solution 2.7

$$E\left[Z^2\right] = \int_0^{20} \left(1{,}000\ v^t\right)^2 f_{T(45)}(t)\,dt + \int_{20}^{\omega-45} \left(1{,}000\ v^{20}\right)^2\ f_{T(45)}(t)\,dt$$

$$= 1{,}000^2 \left(\int_0^{20} e^{-0.10t}\ 0.02\, e^{-0.02t}\ dt\ +\ \left(e^{-0.05\times 20}\right)^2\ {}_{20}p_{45} \right)$$

$$= 1{,}000^2 \left(\frac{0.02\left(1-e^{-2.4}\right)}{0.12}\ +\ e^{-2.4} \right) = 242{,}264.96 \quad \Rightarrow$$

$$\text{var}(Z) = 242{,}264.96 - (461.85)^2\ = 28{,}955$$

Solution 2.8

You should begin by reviewing the derivation of the formula below in Section 2.5.

$$\text{Premium} = \frac{F}{n} = \underbrace{E[Z]}_{\substack{\text{single benefit} \\ \text{premium}}} + \underbrace{z_\alpha \sqrt{\frac{\text{var}(Z)}{n}}}_{\text{risk charge per policy}}$$

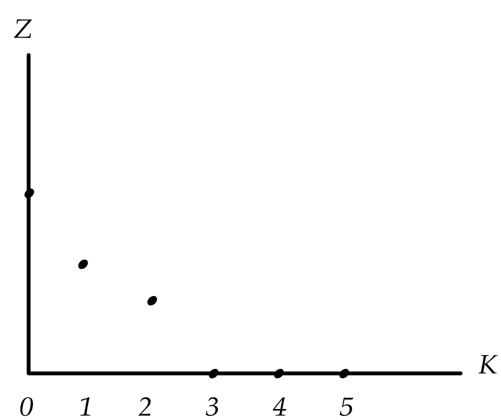
$$= 461.85 + \underbrace{1.282\sqrt{\frac{28{,}955}{100}}}_{21.81}\ =\ 483.67$$

Solution 2.9

$Z = 1{,}000\, v^{K(45)+1}$ for $K(45) = 0$, 1 , 2 , and zero otherwise.

Solution 2.10

If (45) dies at 47.8, the $K(45) = 2$, and $Z = 1{,}000 \times 1.05^{-3} = 863.84$. If (45) dies after age 48 there is no benefit. So the value of Z is zero.



Solution 2.11

The APV symbol is $1{,}000\,A^{1}_{45:\overline{3}|}$. The A denotes actuarial present value for an insurance plan with unit benefit. The 1,000 multiplier converts the face value to 1,000. The lack of an over-bar on the A indicates that a benefit would be paid on the policy anniversary immediately after death. The other element $^{\;1}_{45:\overline{3}|}$ indicates that the benefit is paid if the life (45) expires before the 3-year period expires.

Solution 2.12

The probability function for $K(45)$ is: $\Pr\big(K(45)=k\big) = {}_{k|}q_{45}$. The APV can be written out using the 3-factor method: Sum over all possible times of payment the product of the amount due at a given time, a discount factor from the time of payment, and a probability that the payment occurs.

$$1{,}000\,A^{1}_{45:\overline{3}|} = 1000\cdot v\cdot {}_{0|}q_{45} + 1000\cdot v^2\cdot {}_{1|}q_{45} + 1000\cdot v^3\cdot {}_{2|}q_{45}$$

Solution 2.13

$$\begin{aligned}1{,}000\,A^{1}_{45:\overline{3}|} &= 1000\cdot v\cdot {}_{0|}q_{45} + 1000\cdot v^2\cdot {}_{1|}q_{45} + 1000\cdot v^3\cdot {}_{2|}q_{45}\\ &= 1{,}000\,(1.05^{-1}\times 0.015 + 1.05^{-2}\times(1-0.015)\times 0.018\\ &\qquad + 1.05^{-3}\times(1-0.015)(1-0.018)\times 0.022)\\ &= 48.75\end{aligned}$$

Solution 2.14

We have $Z^2 = 1{,}000\left(v^{K(45)+1}\right)^2 = 1{,}000^2\left(v^2\right)^{K(45)+1}$ for $K(45)=0,1,2$. To compute $E\left[Z^2\right]$ you can simply square the amount and discount factors that appear in the sum for $E[Z]$:

$$\begin{aligned}E\left[Z^2\right] = 1{,}000^2\times {}^2A^{1}_{45:\overline{3}|} &= (1000\cdot v)^2\cdot {}_{0|}q_{45} + \left(1000\cdot v^2\right)^2\cdot {}_{1|}q_{45} + \left(1000\cdot v^3\right)^3\cdot {}_{2|}q_{45}\\ &= 1{,}000^2\,(1.05^{-2}\times 0.015 + 1.05^{-4}\times(1-0.015)\times 0.018\\ &\qquad + 1.05^{-6}\times(1-0.015)(1-0.018)\times 0.022)\\ &= 44{,}071.38\end{aligned}$$

So the variance is:

$$\operatorname{var}(Z) = 44{,}071.38 - (48.75)^2 = 41{,}695$$

Solution 2.15

Since there is no pure endowment, and since the only difference between the discrete model and the continuous model is the timing of the benefit payment, we can use Theorems 1 and 2 to make simple adjustments to the moments calculated for the discrete model in order to find the moments for the corresponding continuous model: $\bar{Z} = 1{,}000\, v^{T(45)}$ if $0 < T(45) \le 3$, zero otherwise.

$$E\left[\bar{Z}\right] = 1{,}000 \bar{A}^{1}_{45:\overline{3|}} = 1{,}000 \times \frac{i}{\delta} \times A^{1}_{45:\overline{3|}} = \frac{0.05}{\ln(1.05)} \times 48.75 = 49.96$$

$$E\left[\bar{Z}^2\right] = 1{,}000^2 \times {}^2\bar{A}^{1}_{45:\overline{3|}} = 1{,}000^2 \times \frac{2i + i^2}{2\delta} \times {}^2A^{1}_{45:\overline{3|}} = \frac{0.1025}{2\ln(1.05)} \times 44{,}071.38 = 46{,}293.31$$

$$\Rightarrow \ \operatorname{var}\left(\bar{Z}\right) = 46{,}293.31 - (49.96)^2 = 43{,}797$$

Solution 2.16

The benefit pattern associated with $50(IA)_{45}$ is 50, 100, 150, … , and so on. Augment this pattern with a level amount of 950 and you have the desired benefit pattern. So the APV for this insurance is:

$$50(IA)_{45} + 950\, A_{45}$$

Solution 2.17

$$\bar{Z} = \begin{cases} 50{,}000 v^{T(45)} & \text{if } T(45) \le 5 \\ 100{,}000 v^{T(45)} & \text{if } T(45) > 5 \end{cases} \quad \Rightarrow$$

$$E\left[\bar{Z}\right] = \int_0^5 50{,}000\, v^t\, f_{T(45)}(t)\, dt + \int_5^{\omega-45} 100{,}000\, v^t\, f_{T(45)}(t)\, dt$$

$$E\left[\bar{Z}^2\right] = \int_0^5 \left(50{,}000\, v^t\right)^2 f_{T(45)}(t)\, dt + \int_5^{\omega-45} \left(100{,}000\, v^t\right)^2 f_{T(45)}(t)\, dt$$

Solution 2.18

The mortality law is de Moivre's law with $\omega = 90$. So the future lifetime after age 45 is uniformly distributed on the interval $[0, 45]$: $f_{T(45)}(t) = 1/45$ for $0 < t \le 45$.

$$E\left[\bar{Z}\right] = \int_0^5 50{,}000\, v^t\, f_{T(45)}(t)\, dt + \int_5^{\omega-45} 100{,}000\, v^t\, f_{T(45)}(t)\, dt$$

$$= 50{,}000 \left(\int_0^5 \frac{1.05^{-t}}{45} dt + \int_5^{45} \frac{2 \times 1.05^{-t}}{45} dt \right)$$

$$= 50{,}000 \left(\frac{1 - 1.05^{-5}}{45 \ln(1.05)} + \frac{2\left(1.05^{-5} - 1.05^{-45}\right)}{45 \ln(1.05)} \right) = 35{,}548$$

Solution 2.19

There is a subtle point here that allows an easy solution. The mortality model is linear. As a result it satisfies the UDD assumption, which only requires that the life table function be piecewise linear. Since the plan of insurance has no pure endowment, we can apply Theorem 1. Let $\overline{Z}$ be the RPV (random present value) if the benefit is paid at death, and let Z be the RPV if the benefit is paid on the policy anniversary immediately following death. By Theorem 1, we have:

$$E\left[\overline{Z}\right] = \frac{i}{\delta} E[Z]$$

We saw in Solution 2.18 that $E\left[\overline{Z}\right] = 35,548$. As a result we have:

$$E[Z] = \frac{\delta}{i} E\left[\overline{Z}\right] = \frac{\ln(1.05)}{0.05} \times 35,548 = 34,688$$

Solution 2.20

We assume that the benefit amount starts at 50,000 and increases continuously at the compounded force of 1.5% per year until time 5. Then the 50,000 gets doubled to 100,000, the past inflation increases are retained (so that at time 5, the benefit amount is $100,000e^{0.015\times 5}$), and the future increases are also continuous and applied using a compound force of 1.5% per year. So we have:

$$\overline{Z} = \begin{cases} \left(50,000\, e^{0.015T(45)}\right) v^{T(45)} & \text{if } T(45) < 5 \\ \left(100,000 e^{0.015T(45)}\right) v^{T(45)} & \text{if } T(45) \geq 5 \end{cases}$$

and hence:

$$E\left[\overline{Z}\right] = \int_0^5 50,000\, e^{0.015t}\ v^t\ f_{T(45)}(t)\,dt \ + \ \int_5^{\omega-45} 100,000\, e^{0.015t}\ v^t\ f_{T(45)}(t)\,dt$$

$$E\left[\overline{Z}^2\right] = \int_0^5 \left(50,000\, e^{0.015t}\ v^t\right)^2\ f_{T(45)}(t)\,dt \ + \ \int_5^{\omega-45} \left(100,000\, e^{0.015t}\ v^t\right)^2\ f_{T(45)}(t)\,dt$$

