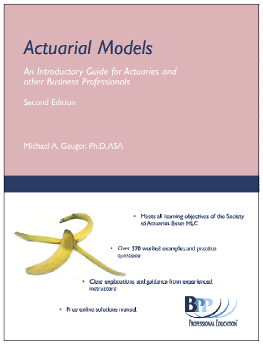

Actuarial Models

Second Edition

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Published by BPP Professional Education

Solutions to practice questions – Chapter 6

Solution 6.1

It means that the time until the second death is more than 10 years. So after 10 years, at least one of the two lives is surviving.

Solution 6.2

It means that the first death must occur within 10 years. This does not exclude the possibility that the second death might also occur within 10 years.

Solution 6.3

With an assumption of constant force, we have ${}_tp_x = {}_tp_y = e^{-0.025t}$.

Question 6.1: probability calculation

$$\Pr\left(T(\overline{50,60}) > 10\right) = 1 - F_{T(\overline{50,60})}(10) = 1 - {}_{10}q_{\overline{50:60}} = 1 - {}_{10}q_{50}\ {}_{10}q_{60} \qquad \text{(independence)}$$

$$= 1 - (1 - {}_{10}p_{50})(1 - {}_{10}p_{60}) = 1 - \left(1 - e^{-0.25}\right)^2 = 0.95107$$

Question 6.2: probability calculation

$$\Pr\left(T(50,60) \le 10\right) = F_{T(50,60)}(10) = 1 - {}_{10}p_{50:60} = 1 - {}_{10}p_{50}\ {}_{10}p_{60} \qquad \text{(independence)}$$

$$= 1 - \left(e^{-0.25}\right)^2 = 0.39347$$

Solution 6.4

The additive probability law results in the following:

$$\begin{aligned} {}_tp_{\overline{xy}} &= \Pr\left(\{T(x) > t\} \cup (T(y) > t)\right) \\ &= \Pr\left(T(x) > t\right) + \Pr\left(T(y) > t\right) - \Pr\left(\{T(x) > t\} \cap (T(y) > t)\right) \\ &= {}_tp_x + {}_tp_y - {}_tp_{xy} = \underbrace{{}_tp_x + {}_tp_y - {}_tp_x\ {}_tp_y}_{\text{with independence}} \end{aligned}$$

Solution 6.5

From the force formulas that are given you can compute the survival functions:

$$
\begin{aligned}
{}_t p_x &= \exp\left(-\int_0^t \mu_x(s)\,ds\right) = \exp\left(-\int_0^t \frac{1}{30-s}\,ds\right) \qquad \text{for } 0 \le t \le 30 \\
&= \exp\left(\ln(30-s)\Big|_0^t\right) = \exp\left(\ln(30-t) - \ln(30)\right) = \frac{30-t}{30} \quad \text{(de Moivre's law)}
\end{aligned}
$$

Similarly, we have ${}_t p_y = e^{-0.03t}$ for all $t > 0$. We have the following life expectancies:

$$
\begin{aligned}
\mathring{e}_x &= \int_0^{30} {}_t p_x\,dt = \int_0^{30} \frac{30-t}{30}\,dt = \frac{900 - 30^2/2}{30} = 15 \\
\mathring{e}_y &= \int_0^{\infty} {}_t p_y\,dt = \int_0^{\infty} e^{-0.03t}\,dt = -\frac{e^{-0.03t}}{0.03}\Bigg|_0^{\infty} = -\frac{0-1}{0.03} = 33.33333 \\
\mathring{e}_{xy} &= \int_0^{\infty} {}_t p_x\, {}_t p_y\,dt = \int_0^{30} \frac{(30-t)e^{-0.03t}}{30}\,dt = \frac{1-e^{-0.90}}{0.03} + \left(\frac{(1+0.03t)e^{-0.03t}}{0.03^2 \times 30}\Bigg|_0^{30}\right) \\
&= 19.78101 - 8.42658 = 11.35443 \\
\Rightarrow \mathring{e}_{\overline{xy}} &= \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy} = 36.97890
\end{aligned}
$$

Solution 6.6

$$
\begin{aligned}
\Pr\left(10 < T(\overline{x,y}) \le 20\right) &= {}_{20}q_{\overline{xy}} - {}_{10}q_{\overline{xy}} = {}_{20}q_x\, {}_{20}q_y - {}_{10}q_x\, {}_{10}q_y \\
&= \left(1 - {}_{20}p_x\right)\left(1 - {}_{20}p_y\right) - \left(1 - {}_{10}p_x\right)\left(1 - {}_{10}p_y\right) \\
&= \left(1 - \frac{10}{30}\right)\left(1 - e^{-0.60}\right) - \left(1 - \frac{20}{30}\right)\left(1 - e^{-0.30}\right) \\
&= 0.21440
\end{aligned}
$$

Solution 6.7

Due to independence of the lifetimes and the fact that ${}_t p_x = 0$ for $t > 30$, we have:

$$
\begin{aligned}
f_{T(x,y)}(t) &= {}_t p_{xy}\, \mu_{xy}(t) = {}_t p_x\, {}_t p_y \left(\mu(x+t) + \mu(y+t)\right) \\
&= \frac{30-t}{30} \times e^{-0.03t}\left(\frac{1}{30-t} + 0.03\right) \quad \text{for } 0 \le t < 30
\end{aligned}
$$

Solution 6.8

$$
\begin{aligned}
1{,}000\,\bar{A}_{xy} &= 1{,}000\int_0^{30} e^{-0.05t}\, f_{T(x,y)}(t)\,dt \\
&= 1{,}000\int_0^{30} e^{-0.05t}\,\frac{30-t}{30}\times e^{-0.03t}\left(\frac{1}{30-t}+0.03\right)dt \\
&= \frac{1{,}000}{30}\left(\int_0^{30} e^{-0.08t}\left(1+0.03(30-t)\right)dt\right) \\
&= \frac{1{,}000}{30}\left(1.9\int_0^{30} e^{-0.08t}\,dt - 0.03\int_0^{30} te^{-0.08t}\,dt\right) \\
&= \frac{1{,}000}{30}\left(1.9\left(\frac{1-e^{-2.4}}{0.08}\right)+0.03\left(\left.\frac{(1+0.08t)\,e^{-0.08t}}{0.08^2}\right|_0^{30}\right)\right) \\
&= \frac{1{,}000}{30}\left(21.59545 - 3.24168\right) = 611.79220
\end{aligned}
$$

Solution 6.9

$$
P\,\bar{a}_{xy} = 1{,}000\,\bar{A}_{xy} \;\Rightarrow\; P = \frac{1{,}000\,\bar{A}_{xy}}{\bar{a}_{xy}} = \frac{1{,}000\,\bar{A}_{xy}}{\left(1-\bar{A}_{xy}\right)/\delta} = 78.79700
$$

Solution 6.10

$$
\begin{aligned}
1{,}000\,\bar{A}_{\overline{xy}} &= 1{,}000\bar{A}_x + 1{,}000\bar{A}_y - 1{,}000\bar{A}_{xy} \\
&= \underbrace{\frac{1{,}000\,\bar{a}_{\overline{30}|}}{30}}_{\substack{\text{de Moivre}\\ \text{law shortcut}}} + \underbrace{\frac{1{,}000\mu}{\mu+\delta}}_{\substack{\text{constant force}\\ \text{shortcut}}} - \underbrace{611.79220}_{\text{Solution 6.9}} \\
&= 517.91323 + 375 - 611.79220 = 281.12103
\end{aligned}
$$

Solution 6.11

The symbol $1{,}000\,\bar{A}^{\;\;1}_{50:60:\overline{10}|}$ is the APV for a continuous joint life 10-year term insurance of 1,000 on a pair of lives currently aged 50 and 60. The benefit of 1,000 is payable immediately at the time the status fails (*ie* at the time of the first death), provided it fails within 10 years.

Solution 6.12

$$
1{,}000\,\bar{A}^{\;\;1}_{50:60:\overline{10}|} = \int_0^{10} 1{,}000\, e^{-\delta t}\; {}_tp_{50}\; {}_tp_{60}\left(\mu(50+t) + \mu(60+t)\right)dt
$$

Solution 6.13

The symbol $1{,}000\,\bar{A}^{\;\;1}_{\overline{50:60}:\overline{10|}}$ is the APV of a continuous last survivor, 10-year term insurance of 1,000. The benefit is payable at the time of the second death, provided that it occurs within 10 years.

Solution 6.14

The relation is:

$$1{,}000\bar{A}^{1}_{50:\overline{10|}} + 1{,}000\bar{A}^{1}_{60:\overline{10|}} = 1{,}000\,\bar{A}^{\;\;1}_{\widehat{50:60}:\overline{10|}} + 1{,}000\,\bar{A}^{\;\;1}_{\overline{50:60}:\overline{10|}}$$

There is a simple intuitive explanation of the relation. The APV's on the left side would purchase benefits of 1,000 at the deaths of 50 and 60, provided they occur within 10 years. The right side would purchase 1,000 at the first death and 1,000 at the second death, provided that they occur within 10 years. Since the benefit patterns are the same, the APV's must be the same.

You could also construct a mathematically precise proof based on the following:

$$\left\{T(50),T(60)\right\} = \left\{T(50,60),T\left(\overline{50,60}\right)\right\} \quad \text{and}$$

$$Z_1 = \begin{cases} 1{,}000e^{-\delta T(50)} & \text{if } T(50) \le 10 \\ 0 & \text{otherwise} \end{cases}$$

$$Z_2 = \begin{cases} 1{,}000e^{-\delta T(60)} & \text{if } T(60) \le 10 \\ 0 & \text{otherwise} \end{cases}$$

$$Z_3 = \begin{cases} 1{,}000e^{-\delta T(50,60)} & \text{if } T(50,60) \le 10 \\ 0 & \text{otherwise} \end{cases}$$

$$Z_4 = \begin{cases} 1{,}000e^{-\delta T\left(\overline{50,60}\right)} & \text{if } T\left(\overline{50,60}\right) \le 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad Z_1 + Z_2 = Z_3 + Z_4$$

Now take an expected value of this final equation to produce the desired relation.

Solution 6.15

$$q_x = 0.025 \ , \ q_{x+1} = 0.035 \ , \ i = 0.06 \ \Rightarrow$$

$$q_{xx} = 1 - p_{xx} = 1 - \left(p_x\right)^2 = 0.04938$$

$${}_{1|}q_{xx} = p_{xx} - {}_2p_{xx} = \left(p_x\right)^2 - \left({}_2p_x\right)^2 = 0.06538$$

$$1{,}000\,A^{1}_{\widehat{xx}:\overline{2|}} = 1{,}000\left(v\,q_{xx} + v^2\,{}_{1|}q_{xx}\right) = 104.76747$$

Solution 6.16

$$\ddot{a}_{x:x:\overline{2}|} = 1 + vp_{xx} = 1 + \frac{0.975^2}{1.06} = 1.89682$$

$$P\,\ddot{a}_{x:x:\overline{2}|} = 1{,}000\,A^{\;1}_{\overset{\frown}{xx}:\overline{2}|} \;\Rightarrow\; P = \frac{104.76747}{1.89682} = 55.23333$$

Solution 6.17

Event	Loss function value	Probability
$K(x,x)=0$	$1{,}000/1.06 - 55.23333 = 888.16289$	$q_{xx} = 0.04938$
$K(x,x)=1$	$1{,}000/1.06^2 - 55.23333(1+v) = 782.65619$	${}_{1\|}q_{xx} = 0.06538$
$K(x,x) \geq 2$	$0 - 55.23333(1+v) = -107.34025$	${}_2p_{xx} = 0.88525$

The expected loss function is zero, so the variance is the same as the second moment:

$$\begin{aligned} \text{var}(L) &= E\left[L^2\right] - \left(E[L]\right)^2 = E\left[L^2\right] - 0 \\ &= (888.16289)^2 \times 0.04938 + (782.65619)^2 \times 0.06538 + (-107.34025)^2 \times 0.88525 \\ &= 89{,}196 \end{aligned}$$

Solution 6.18

$$\begin{aligned} {}_{10}q_{xy} &= {}_{10}q_x + {}_{10}q_y - {}_{10}q_{\overline{xy}} \\ &= F_{T(x),T(y)}(10,\,30) + F_{T(x),T(y)}(30,\,10) - F_{T(x),T(y)}(10,\,10) \\ &= 0.45534 + 0.45534 - 0.22222 = 0.68846 \end{aligned}$$

Solution 6.19

With de Moivre's law we have $l_x = 90 - x$, so the appropriate survival functions are:

$${}_tp^*_{50} = \frac{40-t}{40} \text{ for } t \leq 40 \qquad {}_tp^*_{60} = \frac{30-t}{30} \text{ for } t \leq 30$$

$${}_tp^*_{50:60} = {}_tp^*_{50}\,{}_tp^*_{60} = \frac{1{,}200 - 70t + t^2}{1{,}200} \quad \text{for } t \leq 30$$

$${}_tp_{50:60} = {}_tp^*_{50}\,{}_tp^*_{60}\,e^{-\lambda t} = \frac{1{,}200 - 70t + t^2}{1{,}200} \times e^{-0.02t} \quad \text{for } t \leq 30$$

$${}_{10}q_{50:60} - {}_{10}q^*_{50:60} = {}_{10}p^*_{50:60} - {}_{10}p_{50:60} = 0.50000 - 0.40937 = 0.09063$$

Solution 6.20

$l_x = 90 - x$

${}_t p_{50} = \frac{40-t}{40}$ for $t \le 40$ and 0 otherwise

${}_t q_{60} = 1 - \frac{l_{60+t}}{l_{60}} = \frac{t}{30}$ for $t \le 30$ and 1 otherwise

$$\Rightarrow \; {}_t p_{50} \; {}_t q_{60} = \begin{cases} \frac{40t - t^2}{1,200} & \text{if } t \le 30 \\ \frac{40-t}{40} & \text{if } 30 \le t \le 40 \\ 0 & \text{if } t > 40 \end{cases}$$

$$1,000 \, {}_{(60)|}\bar{a}_{50} = \int_0^\infty \underbrace{1.10^{-t}}_{\text{discount}} \times \underbrace{{}_t p_{50} \; {}_t q_{60}}_{\text{probability}} \times \underbrace{1,000 \, dt}_{\text{amount}} \quad \text{(current payment method)}$$

$$= 1,000 \int_0^{30} 1.10^{-t} \times \frac{40t - t^2}{1,200} dt + 1,000 \int_{30}^{40} 1.10^{-t} \times \frac{40-t}{40} dt$$

