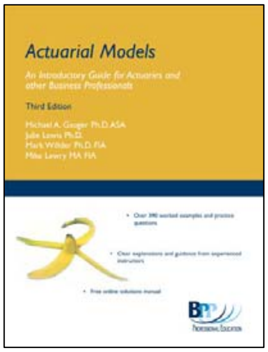

Actuarial Models

Third Edition

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Solutions to practice questions – Chapter 10

Solution 10.1

The rate of this process is $\lambda = 0.2$ per minute. So the number of coins found during a 2-minute walk follows a Poisson distribution with mean $2\lambda = 0.40$. As a result, we have:

$$\Pr\left(N(2) \geq 2\right) = 1 - e^{-0.4}\left(1 + 0.4\right) = 0.06155$$

Solution 10.2

$$N(2) \sim \text{Poisson mean } 0.4 \quad \Rightarrow \quad E[N] = \operatorname{var}(N) = 0.4$$

Solution 10.3

The distribution of $N(5)$ is Poisson with parameter $5\lambda = 1$.

$$\left\{N(5) \geq 2 \text{ and } N(10) \geq 3\right\} = \left\{N(5) = 2 \text{ and } N(10) - N(5) \geq 1\right\} \cup \left\{N(5) \geq 3\right\}$$

$$\begin{aligned}
\Pr\left(N(5) \geq 2 \text{ and } N(10) \geq 3\right) &= \Pr\left(N(5) = 2 \text{ and } N(10) - N(5) \geq 1\right) + \Pr\left(N(5) \geq 3\right) \\
&= \Pr\left(N(5) = 2\right) \underbrace{\Pr\left(N(10) - N(5) \geq 1\right)}_{\text{same as } \Pr\left(N(5) \geq 1\right)} + \Pr\left(N(5) \geq 3\right) \\
&= e^{-1}\frac{1^2}{2!} \times \left(1 - e^{-1}\right) + \left(1 - e^{-1}\left(1 + 1 + \frac{1^2}{2!}\right)\right) = 0.19657
\end{aligned}$$

Solution 10.4

$$\Pr\left(N(5)=2 \mid N(10)=3\right) = \frac{\Pr\left(N(5)=2 \text{ and } N(10)=3\right)}{\Pr\left(N(10)=3\right)}$$

$$= \frac{\Pr\left(N(5)=2 \text{ and } N(10)-N(5)=1\right)}{\Pr\left(N(10)=3\right)} = \frac{\Pr\left(N(5)=2\right)\Pr\left(N(10)-N(5)=1\right)}{\Pr\left(N(10)=3\right)}$$

$$= \frac{\Pr\left(N(5)=2\right)\Pr\left(N(5)=1\right)}{\Pr\left(N(10)=3\right)} = \frac{e^{-1}\frac{1^2}{2!} \times e^{-1}\frac{1^1}{1!}}{e^{-2}\frac{2^3}{3!}} = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{3}{8}$$

Solution 10.5

Since the process rate is $\lambda = 0.4$ per day, the inter-arrival time for this process, T, is exponentially distributed with mean $1/\lambda = 2.5$ days. You are asked to calculate $\Pr(T \le 7 \mid T > 5)$. Due to the memory-less property of the exponential distribution, we have:

$$\Pr(T \le 7 \mid T > 5) = \Pr(T - 5 \le 2 \mid T > 5) = \Pr(T \le 2) = \int_0^2 f_T(t)\,dt = \int_0^2 0.4e^{-0.4t}\,dt = 1 - e^{-0.8} = 0.55067$$

Solution 10.6

We must solve the inequality:

$$0.90 \le \Pr\left(N(n) \ge 2\right) = 1 - e^{-0.4n}\left(1 + 0.4n\right)$$

By trial and error you will find that the right hand side is 0.874 when $n = 9$, and it is 0.908 when $n = 10$. So the intersection must be observed for 10 full days to have at least a 90% chance of seeing at least 2 accidents.

Solution 10.7

The generator matrix for this model is:

$$A = \begin{pmatrix} -0.4 & 0.1 & 0.3 \\ 0 & -0.2 & 0.2 \\ 0 & 0 & 0 \end{pmatrix}$$

So the matrix form of the forward equations can be written:

$$\frac{\partial}{\partial t}\begin{pmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) \\ p_{21}(t) & p_{22}(t) & p_{23}(t) \\ p_{31}(t) & p_{32}(t) & p_{33}(t) \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) \\ p_{21}(t) & p_{22}(t) & p_{23}(t) \\ p_{31}(t) & p_{32}(t) & p_{33}(t) \end{pmatrix}\begin{pmatrix} -0.4 & 0.1 & 0.3 \\ 0 & -0.2 & 0.2 \\ 0 & 0 & 0 \end{pmatrix}$$

Pulling out the relevant matrix entry, we find that:

$$\frac{\partial}{\partial t}p_{11}(t) = -0.4p_{11}(t)$$

and the solution of this differential equation (subject to the correct initial condition that $p_{11}(0)=1$) is $p_{11}(t)=e^{-0.4t}$.

Alternatively, we can observe that since it is never possible to re-enter state 1 once the state has been left, $p_{11}(t)$ *must be the same as the occupancy probability* $p_{\overline{11}}(t)$*. Since the model is time-homogeneous, the waiting time is exponential and the holding time probability is just* $1-F(t)$*, where* $F(t)$ *is the distribution function of an exponential distribution with mean equal to the sum of the transition rates out of the relevant state.*

Solution 10.8

To find $p_{12}(t)$, we can use the matrix equation from the previous question. Again pulling out the relevant matrix entry, we find that:

$$\frac{\partial}{\partial t} p_{12}(t) = 0.1 p_{11}(t) - 0.2 p_{12}(t)$$

Substituting in the expression for $p_{11}(t)$ that we found in the previous question:

$$\frac{\partial}{\partial t} p_{12}(t) = 0.1 e^{-0.4t} - 0.2 p_{12}(t)$$

Rearranging this differential equation:

$$\frac{\partial}{\partial t} p_{12}(t) + 0.2 p_{12}(t) = 0.1 e^{-0.4t}$$

Multiplying through by the integrating factor of $e^{0.2t}$ we get:

$$\frac{\partial}{\partial t} p_{12}(t) e^{0.2t} + 0.2 e^{0.2t} p_{12}(t) = 0.1 e^{-0.2t}$$

Integrating both sides of this equation:

$$e^{0.2t} p_{12}(t) = \int 0.1 e^{-0.2t} dt = -0.5 e^{-0.2t} + C$$

To find the relevant constant, we observe that $p_{12}(0) = 0$, so that $C = 0.5$, and that $p_{12}(t) = 0.5\left(e^{-0.2t} - e^{-0.4t}\right)$.

Solution 10.9

The corresponding forward equation for $p_{13}(t)$ is:

$$\frac{\partial}{\partial t} p_{13}(t) = 0.3 p_{11}(t) + 0.2 p_{12}(t)$$

Since we have expressions for both of the terms on the right hand side, we can write:

$$\frac{\partial}{\partial t} p_{13}(t) = 0.3 e^{-0.4t} + 0.2\left[0.5 e^{-0.2t} - 0.5 e^{-0.4t}\right] = 0.2 e^{-0.4t} + 0.1 e^{-0.2t}$$

Integrating this expression directly, we find that:

$$p_{13}(t) = -0.5 e^{-0.4t} - 0.5 e^{-0.2t} + C$$

Using the initial condition that $p_{13}(0) = 0$, we find that $C = 1$, so that $p_{13}(t) = 1 - 0.5 e^{-0.4t} - 0.5 e^{-0.2t}$.

There are a number of comments that could be made about these results.

(1) Note that as $t \to \infty$, $p_{11}(t) \to 0$, $p_{12}(t) \to 0$ and $p_{13}(t) \to 1$. This is consistent with our understanding of the model, since, if we look at the transition diagram, we see that the model is certain to end up in state 3, the absorbing state, in the long term.

(2) We could have found the expression for $p_{13}(t)$ more quickly by using the result $p_{13}(t) = 1 - p_{11}(t) - p_{12}(2)$, rather than by integrating again. Since the model must be in one of the three states at any time t, the sum of the three probabilities is 1, for any value of t.

Solution 10.10

Since it is not possible to return to state 1 once the state has been left, we have:

$$p_{11}(t) = p_{\overline{11}}(t) = e^{-(\alpha+\beta)t}$$

For $p_{12}(t)$ we need the Kolmogorov forward differentialequation. In matrix form, we have:

$$\frac{\partial}{\partial t}\begin{pmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) \\ p_{21}(t) & p_{22}(t) & p_{23}(t) \\ p_{31}(t) & p_{32}(t) & p_{33}(t) \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) \\ p_{21}(t) & p_{22}(t) & p_{23}(t) \\ p_{31}(t) & p_{32}(t) & p_{33}(t) \end{pmatrix}\begin{pmatrix} -(\alpha+\beta) & \alpha & \beta \\ 0 & -\gamma & \gamma \\ 0 & 0 & 0 \end{pmatrix}$$

Selecting the relevant entry:

$$\frac{\partial}{\partial t} p_{12}(t) = \alpha p_{11}(t) - \gamma p_{12}(t)$$

Substituting in the relevant expression for $p_{11}(t)$ and rearranging:

$$\frac{\partial}{\partial t} p_{12}(t) + \gamma p_{12}(t) = \alpha e^{-(\alpha+\beta)t}$$

Now multiplying through by the integrating factor $e^{\gamma t}$:

$$e^{\gamma t}\frac{\partial}{\partial t} p_{12}(t) + \gamma e^{\gamma t} p_{12}(t) = \alpha e^{-(\alpha+\beta-\gamma)t}$$

Integrating both sides of this equation:

$$e^{\gamma t} p_{12}(t) = \int \alpha e^{-(\alpha+\beta-\gamma)t}\,dt = \frac{-\alpha}{\alpha+\beta-\gamma} e^{-(\alpha+\beta-\gamma)t} + C$$

Noting that when $t=0$ we have $p_{12}(0)=0$, we find that $C = \dfrac{\alpha}{\alpha+\beta-\gamma}$, and so:

$$p_{12}(t) = \frac{\alpha}{\alpha+\beta-\gamma}\left[e^{-\gamma t} - e^{-(\alpha+\beta)t}\right]$$

Solution 10.11

We can find $p_{13}(t)$ by subtraction:

$$p_{13}(t) = 1 - p_{11}(t) - p_{12}(t) = 1 - e^{-(\alpha+\beta)t} - \frac{\alpha}{\alpha+\beta-\gamma}\left[e^{-\gamma t} - e^{-(\alpha+\beta)t}\right]$$

Solution 10.12

We have:

$$p_{11}(0,4)=p_{11}(0,1)p_{11}(1,4)$$

For $t<1$ we have $\alpha+\beta=0.6$, and for $t>1$ we have $\alpha+\beta=0.3$. So the overall probability is:

$$p_{11}(0,4)=e^{-0.6\times 1-0.3\times 3}=e^{-1.5}=0.22313$$

For $p_{12}(0,4)$, using the same logic, we have:

$$p_{12}(0,4)=p_{11}(0,1)p_{12}(1,4)+p_{12}(0,1)p_{22}(1,4)$$

We now need an expression for $p_{22}(t)$.

Going back to the generator matrix, we see that:

$$\frac{\partial}{\partial t}p_{22}(t)=\alpha p_{21}(t)-\gamma p_{22}(t)$$

But, using this model, it is not possible to return to state 1 from state 2. So $p_{21}(t)=0$ for all t, and the equation becomes:

$$\frac{\partial}{\partial t}p_{22}(t)=-\gamma p_{22}(t) \quad \Rightarrow \quad p_{22}(t)=e^{-\gamma t}$$

For $t>1$, we have $\gamma=0.2$. So:

$$p_{22}(1,4)=e^{-0.2\times 3}=e^{-0.6}$$

So we can now calculate the value of $p_{12}(0,4)$:

$$\begin{aligned}p_{12}(0,4)&=p_{11}(0,1)p_{12}(1,4)+p_{12}(0,1)p_{22}(1,4)\\&=e^{-0.6}\times\frac{0.2}{0.2+0.1-0.2}\left[e^{-0.2\times 3}-e^{-0.3\times 3}\right]+\frac{0.4}{0.5+0.2-0.1}\left(e^{-0.1\times 1}-e^{-0.6\times 1}\right)\times e^{-0.6}\\&=0.156128+0.156313\\&=0.31244\end{aligned}$$

Solution 10.13

We can now write:

$$p_{13}(0,4) = p_{11}(0,1)p_{13}(1,4) + p_{12}(0,1)p_{23}(1,4) + p_{13}(0,1)p_{33}(1,4)$$

Note first that $p_{33}(1,4) = 1$ since this is the absorbing state. We now need $p_{23}(1,4)$. But $p_{23}(t) = 1 - p_{22}(t) = 1 - e^{-\gamma t}$, so that:

$$p_{23}(1,4) = 1 - e^{-0.6}$$

We can now put together the whole expression for $p_{13}(0,4)$:

$$\begin{aligned} p_{13}(0,4) &= p_{11}(0,1)p_{13}(1,4) + p_{12}(0,1)p_{23}(1,4) + p_{13}(0,1)p_{33}(1,4) \\ &= e^{-0.6}\left[1 - e^{-0.9} - 2\left(e^{-0.6} - e^{-0.9}\right)\right] + 0.8\left[e^{-0.1} - e^{-0.6}\right]\left(1 - e^{-0.6}\right) + \left[1 - e^{-0.6} - 0.8\left(e^{-0.1} - e^{-0.6}\right)\right] \\ &= 0.169553 + 0.128508 + 0.166368 \\ &= 0.46443 \end{aligned}$$

Solution 10.14

The APV of the benefit is:

$$\begin{aligned} &100\int_0^\infty e^{-\delta t}\, p_{\overline{11}}(t)\left(\mu_{12} + \mu_{13}\right) dt \\ &= 100\int_0^\infty e^{-0.04t}\, e^{-0.06t}\,(0.05 + 0.01)\, dt \\ &= 6\int_0^\infty e^{-0.10t}\, dt \\ &= \frac{6}{0.1} = 60 \end{aligned}$$

Solution 10.15

The APV of the benefit valued at the time of entering state 2 is:

$$10\int_0^\infty e^{-\delta t}\, p_{\overline{22}}(t)\, dt = 10\int_0^\infty e^{-0.04t}\, e^{-0.03t} dt = 10\int_0^\infty e^{-0.07t}\, dt = \frac{10}{0.07} = \frac{1,000}{7}$$

So the APV of the benefit valued at the date of issue of the policy is:

$$\frac{1,000}{7}\int_0^\infty e^{-\delta t}\, p_{\overline{11}}(t)\,\mu_{12}\, dt = \frac{1,000}{7}\int_0^\infty e^{-0.04t}\, e^{-0.06t}\,(0.05)\, dt = \frac{50}{7}\int_0^\infty e^{-0.10t} = \frac{50}{7 \times 0.1} = \frac{500}{7} = 71.4286$$

Solution 10.16

The holding time in state 1 is exponentially distributed with mean $\dfrac{1}{0.05 + 0.01} = 16.67$ years.

Solution 10.17

From Solution 10.16, the expected holding time in state 1 is 16.67 years. When the policyholder leaves state 1, he enters state 2 with probability $\frac{0.05}{0.05+0.01}=\frac{5}{6}$ and he enters state 3 with probability $\frac{1}{6}$. Note that these probabilities are just ratios of the forces of transition. $\frac{5}{6}$ ths of the total force out of state 1 is into state 2 and the other $\frac{1}{6}$ is into state 3.

If the policyholder enters state 2, the expected time until he then enters state 3 is $\frac{1}{0.03}=33.33$ years. So the expected time until a new policyholder enters state 3 is:

$$16.67+\frac{5}{6}\times 33.33+\frac{1}{6}\times 0=44.44 \text{ years}$$

Solution 10.18

The continuously payable premium is the solution of the equation:

$$P\bar{a}_x = 100{,}000\bar{A}_x^A + 50{,}000\bar{A}_x^{NA}$$

where $\bar{A}_x^A$ denotes the EPV of a benefit of 1 unit paid on accidental death, and $\bar{A}_x^{NA}$ denotes the EPV of a benefit of 1 unit paid on non-accidental death.

Using the standard formulae for actuarial functions using a constant force of mortality:

$$\bar{a}_x = \frac{1}{\mu+\delta} = \frac{1}{0.005+0.02+0.035} = \frac{1}{0.06} = 16.6667$$

and: $$\bar{A}_x^A = \frac{\mu}{\mu+\delta} = \frac{0.005}{0.005+0.035} = 0.125\,,\ \bar{A}_x^{NA} = \frac{\mu}{\mu+\delta} = \frac{0.02}{0.02+0.035} = 0.36364$$

We have:

$$16.6667P = 100{,}000\times 0.125 + 50{,}000\times 0.36364 \quad \Rightarrow \quad P = 1{,}840.91$$

Solution 10.19

We know that Thiele's equation for this policy is:

$$\frac{\partial}{\partial t}\,{}_tV = {}_tV\delta + P - \left(B_2 - {}_tV\right)\mu_{12}(t) - \left(B_3 - {}_tV\right)\mu_{13}(t)$$

Substituting in the numerical values, we have:

$$\frac{\partial}{\partial t}\,{}_tV = 0.035\,{}_tV + 1,840.91 - (100,000 - {}_tV)0.005 - (50,000 - {}_tV)0.02$$

or: $$\frac{\partial}{\partial t}\,{}_tV = 340.91 + 0.06\,{}_tV$$

Rearranging:

$$\frac{\partial}{\partial t}\,{}_tV - 0.06\,{}_tV = 340.91$$

Multiplying through by the integrating factor $e^{-0.06t}$:

$$\frac{\partial}{\partial t}\,{}_tVe^{-0.06t} - 0.06e^{-0.06t}\,{}_tV = 340.91e^{-0.06t}$$

Integrating both sides of this equation:

$${}_tVe^{-0.06t} = -\frac{340.91}{0.06}e^{-0.06t} + C$$

Setting the reserve at time zero equal to zero, we find that $C = \frac{340.91}{0.06} = 5,681.82$. So we have:

$${}_tV = 5,681.82\left(e^{0.06t} - 1\right)$$

and the reserve at time 10 will be:

$${}_{10}V = 5,681.82\left(e^{0.6} - 1\right) = 4,671.13$$

Solution 10.20

The rate of change of the reserve will be affected by:

(1) interest earned on the reserve: $+\delta\, {}_tV$

(2) the payment of death benefit: $-\mu S$

(3) the release of reserve no longer required on death: $+\mu\, {}_tV$

(4) premiums received: $+P$

So Thiele's equation for this policy is:

$$\frac{\partial}{\partial t}\,{}_tV = \delta\,{}_tV - \mu S + \mu\,{}_tV + P$$

Substituting in the given numerical values:

$$\frac{\partial}{\partial t}\,{}_tV = 0.03\,{}_tV - 0.05 + 0.05\,{}_tV + P$$

So we need to find the premium P. First we have:

$${}_{10}E_x = e^{-0.05\times10}e^{-0.03\times10} = e^{-0.8}$$

So:

$$\bar{a}_{x:\overline{10}|} = \bar{a}_x - {}_{10}E_x\,\bar{a}_{x+10} = \frac{1}{0.08} - e^{-0.8}\times\frac{1}{0.08} = 12.5\left(1-e^{-0.8}\right) = 6.883388$$

Similarly:

$$\bar{A}_{x:\overline{10}|} = \frac{0.05}{0.05+0.03} - e^{-0.8}\times\frac{0.05}{0.05+0.03} + e^{-0.8} = \frac{1}{8}\left(5+3e^{-0.8}\right) = 0.793499$$

So the premium is:

$$P = \frac{\bar{A}_{x:\overline{10}|}}{\ddot{a}_{x:\overline{10}|}} = 0.11528$$

We now have to solve the differential equation:

$$\frac{\partial}{\partial t}\,{}_tV = 0.03\,{}_tV - 0.05 + 0.05\,{}_tV + P$$

Rearranging the equation:

$$\frac{\partial}{\partial t}\,{}_tV - 0.08\,{}_tV = P - 0.05$$

The integrating factor is $e^{-\int 0.08dt} = e^{-0.08t}$. So multiplying through by this, we find that:

$$\frac{\partial}{\partial t}\,{}_tVe^{-0.08t} - 0.08e^{-0.08t}\,{}_tV = (P - 0.05)e^{-0.08t}$$

Integrating both sides of this equation:

$${}_tVe^{-0.08t} = \int (P - 0.05)e^{-0.08t}dt = \frac{P - 0.05}{-0.08}e^{-0.08t} + C$$

Using the initial condition that the reserve will be zero at time zero, we find that $C = \dfrac{P - 0.05}{0.08}$. So the expression for the reserve is:

$${}_tV = \frac{P - 0.05}{0.08}\left(e^{0.08t} - 1\right)$$

Substituting in $P = 0.11528$ and $t = 3$, we find that:

$${}_3V = 0.22133$$

