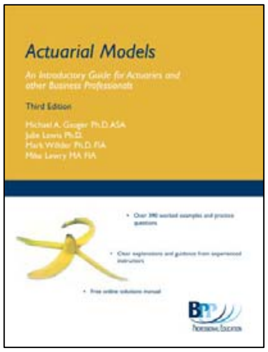

Actuarial Models

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Solutions to practice questions – Chapter 4

Solution 4.1

Benefits		*420 • 5,025*	*441 • 5,025*
Age	*x*	*x*+1	*x*+2
Premium	*10,500P*	*10,080P*	

Solution 4.2

$$10,500\,P + \frac{10,080\,P}{1.05} = \frac{420 \times 5,025}{1.05} + \frac{441 \times 5,025}{1.05^2} \Rightarrow P = \frac{4,020,000}{20,100} = 200$$

Solution 4.3

Year	Beginning fund	Premium	Interest	Benefits	Ending fund
1	0	2,100,000	105,000	2,110,500	94,500
2	94,500	2,016,000	105,525	2,216,025	0

Solution 4.4

If the premium is zero, then the ending fund is calculated as below:

Year	Beginning fund	Premium	Interest	Benefits	Ending fund
1	0	0	0	2,110,500	−2,110,500
2	−2,110,500	0	−105,525	2,216,025	−4,432,050

If the premium is 1, then the ending fund is determined as below:

Year	Beginning fund	Premium	Interest	Benefits	Ending fund
1	0	10,500	525	2,110,500	–2,099,475
2	–2,099,475	10,080	-104,469.75	2,216,025	–4,409,889.75

The linear function $EF(P)$ has slope m:

$$m = \frac{-4,409,889.75 + 4,432,050}{1-0} = 22,160.25$$

Using the slope-intercept form of a linear equation we have:

$$EF(P) = -4,432,050 + 22,160.25P \quad , \; 0 = EF(P) \;\Rightarrow\; P = \frac{4,432,050}{22,160.25} = 200$$

Solution 4.5

$$\mu(x) = \mu = -\ln(0.98) \Rightarrow {}_kp_x = e^{-\mu k} = 0.98^k \; , \; q_{x+k} = 0.02$$

$$A_x = \sum_{k=0}^{\infty} v^{k+1}\,{}_{k|}q_x = \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_x\,q_{x+k} = \sum_{k=0}^{\infty} \frac{1}{1.05^{k+1}} \times 0.98^k \times 0.02$$

$$= \frac{0.02}{1.05}\left(1 + \left(\frac{0.98}{1.05}\right)^1 + \left(\frac{0.98}{1.05}\right)^2 + \cdots\right) \quad \text{(geometric series)}$$

$$= \frac{0.02}{1.05}\left(\frac{1}{1-0.98/1.05}\right) = \frac{0.02}{0.07} \quad \Rightarrow$$

$$\ddot{a}_x = \frac{1-A_x}{d} = \frac{5/7}{5/105} = 15$$

$$1,000\,P_x = \frac{1,000\,A_x}{\ddot{a}_x} = \frac{2,000/7}{15} = 19.04762$$

Solution 4.6

For the semi-continuous plan we have:

$$\bar{A}_x = \frac{\mu}{\mu+\delta} = \frac{-\ln(0.98)}{-\ln(0.98)+\ln(1.05)} = 0.29282 \quad , \quad \ddot{a}_x = 15 \;\; \text{(Solution 4.5)} \;\; \Rightarrow$$

$$1,000P(\bar{A}_x) = \frac{1,000\,\bar{A}_x}{\ddot{a}_x} = \frac{292.82}{15} = 19.52154$$

Solution 4.7

The relation $\bar{A}_x = (i/\delta) A_x$ only holds when the UDD assumption holds. This assumption would require ${}_t p_x$ to be a linear function of *t* when *x* is an integer age and $0 < t < 1$. But with the constant force model we would have ${}_t p_x = 0.98^t$, which is not linear in *t*.

Let's see how close to an answer the UUD law would have given if you had mistakenly applied it:

$$(i/\delta) 1{,}000\, P_x = \left(0.05/\ln(1.05)\right) \times 19.04762 = 19.51994$$

This is pretty close to the true value obtained in Solution 4.6.

Solution 4.8

$$1{,}000\ {}_{10}P\left(\bar{A}^{1}_{30:\overline{35}|}\right) = \frac{1{,}000\, \bar{A}^{1}_{30:\overline{35}|}}{\ddot{a}_{30:\overline{10}|}}$$

Solution 4.9

$$P\, \ddot{a}_{55:\overline{10}|} = 50{,}000\ {}_{10|}\ddot{a}_{55}$$

Solution 4.10

With de Moivre's law it is easy to calculate whole life insurance values from the formula $A_x = a_{\overline{\omega - x}|} / (\omega - x)$:

$$A_{55} = \frac{a_{\overline{35}|}}{35} = \frac{16.37419}{35} = 0.46783 \quad , \quad A_{65} = \frac{a_{\overline{25}|}}{25} = \frac{14.09394}{25} = 0.56376 \quad \Rightarrow$$

$$\ddot{a}_{55} = 11.17548 \quad , \quad \ddot{a}_{65} = 9.16109$$

The premium that we need to calculate is:

$$P = \frac{50{,}000\ {}_{10|}\ddot{a}_{55}}{\ddot{a}_{55:\overline{10}|}} = \frac{50{,}000\ v^{10}\ {}_{10}p_{55}\ \ddot{a}_{65}}{\ddot{a}_{55} - v^{10}\ {}_{10}p_{55}\ \ddot{a}_{65}}$$

$$= \frac{50{,}000 \times (25/35) \times 9.16109 / 1.05^{10}}{11.17548 - (25/35) 9.16109 / 1.05^{10}} = \frac{200{,}861.16}{7.15826} = 28{,}060$$

Solution 4.11

With the first 5 payments guaranteed, the annuity APV ${}_{10|}\ddot{a}_{55}$ is replaced by:

$${}_{10|}\ddot{a}_{\overline{5}|} + {}_{15|}\ddot{a}_{55}$$

So now we need to compute $\ddot{a}_{70}$ as well:

$$A_{70} = \frac{a_{\overline{20}|}}{20} = 0.62311 \quad \Rightarrow \quad \ddot{a}_{70} = 7.91468$$

Finally, the new annual premium amount is:

$$P = \frac{50,000\left(\ddot{a}_{\overline{15}|} - \ddot{a}_{\overline{10}|} + {}_{15|}\ddot{a}_{55}\right)}{\ddot{a}_{55:\overline{10}|}} = \frac{50,000\left(\ddot{a}_{\overline{15}|} - \ddot{a}_{\overline{10}|} + v^{15}\ {}_{15}p_{55}\ \ddot{a}_{70}\right)}{\ddot{a}_{55} - v^{10}\ {}_{10}p_{55}\ \ddot{a}_{65}}$$

$$= \frac{50,000\left(2.79082 + (20/35)\times 7.91468/1.05^{15}\right)}{7.15826} = \frac{248,315.13}{7.15826} = 34,689$$

Solution 4.12

(i) $$1,000\,P^{(2)}\left(\bar{A}_{40}\right) = \frac{1,000\,\bar{A}_{40}}{\ddot{a}^{(2)}_{40}} = \frac{1,000\,A_{40}\,(i/\delta)}{\alpha(2)\ddot{a}_{40} - \beta(2)} = \frac{161.32\left(0.06/\ln(1.06)\right)}{1.00021\times 14.8166 - 0.25739} = 11.40701$$

(ii) $$1,000\,{}_{10}P^{(2)}\left(\bar{A}_{40}\right) = \frac{1,000\,\bar{A}_{40}}{\ddot{a}^{(2)}_{40:\overline{10}|}} = \frac{1,000\,A_{40}\,(i/\delta)}{\alpha(2)\ddot{a}_{40:\overline{10}|} - \beta(2)\left(1 - v^{10}\ {}_{10}p_{40}\right)}$$

$$= \frac{161.32\left(0.06/\ln(1.06)\right)}{1.00021\times\left(14.8166 - 0.53667\times 13.2668\right) - 0.25739\left(1 - 0.53667\right)} = 21.9173$$

Solution 4.13

Apply the equivalence principle. Suppose that P is the level annual premium rate and assume UDD:

$$P\,\ddot{a}_{40} = 1,000\,\bar{A}_{40} + 1,000\,{}_{10|}\bar{A}_{40} \quad \Rightarrow$$

$$P = \frac{1,000\,\bar{A}_{40} + 1,000\,{}_{10|}\bar{A}_{40}}{\ddot{a}_{40}} = \frac{i}{\delta}\times\frac{1,000\,A_{40} + v^{10}\ {}_{10}p_{40}\times 1,000\,A_{50}}{\ddot{a}_{40}}$$

$$= \frac{0.06}{\ln(1.06)}\times\frac{161.32 + 0.53667\times 249.05}{14.8166} = 20.50005$$

Solution 4.14

$$P\left(\ddot{a}_{40} + {}_{10|}\ddot{a}_{40}\right) = 1,000\,\bar{A}_{40} + 1,000\,{}_{10|}\bar{A}_{40} \quad \Rightarrow$$

$$P = \frac{1,000\,\bar{A}_{40} + 1,000\,{}_{10|}\bar{A}_{40}}{\ddot{a}_{40} + {}_{10|}\ddot{a}_{40}} = \frac{i}{\delta}\times\frac{1,000\,A_{40} + v^{10}\ {}_{10}p_{40}\times 1,000\,A_{50}}{\ddot{a}_{40} + v^{10}\ {}_{10}p_{40}\ddot{a}_{50}}$$

$$= \frac{0.06}{\ln(1.06)}\times\frac{161.32 + 0.53667\times 249.05}{14.8166 + 0.53667\times 13.2668} = 13.84638$$

Solution 4.15

Let $K = K(40)$:

$$L = 1{,}000 \times \begin{cases} v^{K+1} & \text{if } K \le 9 \\ v^{10} & \text{if } K \ge 10 \end{cases} - P \times \begin{cases} \ddot{a}_{\overline{K+1|}} & \text{if } K \le 9 \\ \ddot{a}_{\overline{10|}} & \text{if } K \ge 10 \end{cases}$$

$$= \begin{cases} 1{,}000v^{K+1} - P\ddot{a}_{\overline{K+1|}} & \text{if } K \le 9 \\ 1{,}000v^{10} - P\ddot{a}_{\overline{10|}} & \text{if } K \ge 10 \end{cases}$$

Solution 4.16

$$\begin{aligned} E[L] &= 1{,}000\, A_{40:\overline{10|}} - 80\, \ddot{a}_{40:\overline{10|}} \\ &= \left(1{,}000\, A_{40} - v^{10}\,{}_{10}p_{40}\, 1{,}000\, A_{50} + 1{,}000\, v^{10}\,{}_{10}p_{40}\right) - 80\left(\ddot{a}_{40} - v^{10}\,{}_{10}p_{40}\, \ddot{a}_{50}\right) \\ &= 564.34 - 80 \times 7.69671 = -51.40 \end{aligned}$$

Solution 4.17

For an insurance of 1, the variance of the loss function at issue (L_1) using the formula from Section 4.5 of the text:

$$\operatorname{var}(L_1) = \left(1 - E[L_1]\right)^2 \left(\frac{{}^2A_{40:\overline{10|}} - \left(A_{40:\overline{10|}}\right)^2}{\left(1 - A_{40:\overline{10|}}\right)^2} \right)$$

If the benefit were 1 and the premium were 80/1,000, then $E[L_1] = -51.39914/1{,}000 = -0.05140$. Obtain the variance on a unit basis using the formula above. Then multiply by $1{,}000^2$ to obtain variance if the benefit is 1,000.

$$A_{40:\overline{10|}} = 0.56434 \qquad \text{(Solution 4.16)}$$

$$\begin{aligned} {}^2A_{40:\overline{10|}} &= \left({}^2A_{40} - v^{20}\,{}_{10}p_{40}\, {}^2A_{50}\right) + v^{20}\,{}_{10}p_{40} \\ &= \left(0.04863 - \frac{0.53667}{1.06^{10}} \times 0.09476\right) + \frac{0.53667}{1.06^{10}} = 0.31991 \end{aligned}$$

$$\operatorname{var}(L_1) = \left(1 - E[L_1]\right)^2 \frac{{}^2A_{40:\overline{10|}} - \left(A_{40:\overline{10|}}\right)^2}{\left(1 - A_{40:\overline{10|}}\right)^2} = 1.05140^2 \times \left(\frac{0.31991 - 0.56434^2}{(1 - 0.56434)^2} \right) = 0.00833$$

Multiply this variance by $1{,}000^2$: $\operatorname{var}(L) = \operatorname{var}(1{,}000\, L_1) = 1{,}000^2 \times 0.00833 = 8{,}330$

Note: Your answer may differ slightly, depending on the intermediate rounding you've applied

Solution 4.18

With de Moivre's law we have:

$$\bar{A}_{40} = \frac{\bar{a}_{\overline{60}|}}{60} = 0.27019 \quad , \quad \bar{a}_{40} = \frac{1-\bar{A}_{40}}{\delta} = 12.16354 \quad , \quad \bar{P}\left(\bar{A}_{40}\right) = 0.02221$$

The insurer's loss function at issue is:

$$L = e^{-\delta T(40)} - \bar{P}\left(\bar{A}_{40}\right)\bar{a}_{\overline{T(40)}|} = \left(1 + \frac{\bar{P}\left(\bar{A}_{40}\right)}{\delta}\right)e^{-\delta T(40)} - \frac{\bar{P}\left(\bar{A}_{40}\right)}{\delta}$$

The loss is positive if the following relation holds:

$$T(40) < -\frac{1}{\delta}\ln\left(\frac{\bar{P}\left(\bar{A}_{40}\right)}{\bar{P}\left(\bar{A}_{40}\right) + \delta}\right) = 21.81063$$

The probability of this event is:

$$\Pr\left(T(40) < 21.81063\right) = \int_0^{21.81063} f_{T(40)}(t)\,dt = \int_0^{21.81063} \frac{1}{60}\,dt = \frac{21.81063}{60} = 0.36351$$

Solution 4.19

The loss function is a decreasing function of $T(40)$, so it flips the percentiles. Since we are looking for the median value, we merely need to plug $T = t_{0.5}$ into the loss function formula.

The future lifetime is uniformly distributed on the interval $\left[0, 60\right]$. The median value is $t_{0.5} = 30$. So the median value of the loss function is:

$$L = \left(1 + \frac{\bar{P}\left(\bar{A}_{40}\right)}{\delta}\right)e^{-\delta T(40)} - \frac{\bar{P}\left(\bar{A}_{40}\right)}{\delta} = \left(1 + \frac{0.02221}{0.06}\right)e^{-0.06\times 30} - \frac{0.02221}{0.06} = -0.14372$$

Solution 4.20

The annual premium amount is $P = 1/\bar{s}_{\overline{t_{0.2}}|}$ where $t_{0.2}$ is the 20th percentile of future lifetime. (See Example 4.13):

$$0.2 = \Pr\left(T(x) \le t_{0.2}\right) \;\Rightarrow\; 0.8 = {}_{t_{0.2}}p_x = e^{-\mu t_{0.2}} \;\Rightarrow\; t_{0.2} = -\ln(0.8)/\mu$$

$$\bar{s}_{\overline{t_{0.2}}|} = \frac{e^{\delta t_{0.2}} - 1}{\delta} = \frac{e^{-\ln(0.8)\delta/\mu} - 1}{\delta} = \frac{1.25^{\delta/\mu} - 1}{\delta}$$

$$P = \frac{1}{\bar{s}_{\overline{t_{0.2}}|}} = \frac{\delta}{1.25^{\delta/\mu} - 1}$$

