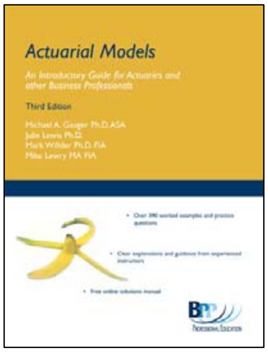

Actuarial Models

Third Edition

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Solutions to practice questions – Chapter 7

Solution 7.1

Survival for a period of time means remaining a member of the group for this period, in other words, avoiding all the modes of decrement.

Solution 7.2

(i) $${}_2p_{26}^{(\tau)} = \frac{l_{28}^{(\tau)}}{l_{26}^{(\tau)}} = \frac{919}{950}$$

(ii) $${}_2q_{25}^{(\tau)} = \frac{d_{25}^{(\tau)} + d_{26}^{(\tau)}}{l_{25}^{(\tau)}} = \frac{50+14}{1{,}000}$$

(iii) $${}_{1|}q_{26}^{(2)} = \frac{d_{27}^{(2)}}{l_{26}^{(\tau)}} = \frac{8}{950}$$

(iv) $${}_2q_{26}^{(1)} = \frac{d_{26}^{(1)} + d_{27}^{(1)}}{l_{26}^{(\tau)}} = \frac{8+9}{950}$$

Solution 7.3

From the formulas for the force functions, we have:

$$\mu^{(i)}(x+t) = \frac{r_i}{20-t} \quad \Rightarrow \quad {}_tp_x'^{(i)} = \exp\left(-\int_0^t \frac{r_i}{20-s}\,ds\right) = \left(\frac{20-t}{20}\right)^{r_i} \quad \text{for } 0 \le t < 20$$

Cause 1 , $r_1 = 1$

$${}_tp_x'^{(1)} = \frac{20-t}{20} \quad , \quad f_{T_1}(t) = {}_tp_x'^{(1)}\,\mu^{(1)}(x+t) = \frac{20-t}{20} \times \frac{1}{20-t} = \frac{1}{20} \quad \text{and}$$

$${}_tq_x'^{(1)} = F_{T_1}(t) = \frac{t}{20} \quad \text{for } 0 \le t < 20$$

Cause 2 , $r_2 = 0.5$

$${}_tp_x'^{(2)} = \left(\frac{20-t}{20}\right)^{0.5} \quad , \quad f_{T_2}(t) = {}_tp_x'^{(2)}\,\mu^{(2)}(x+t) = \left(\frac{20-t}{20}\right)^{0.5} \times \frac{0.5}{20-t} = \frac{1}{\sqrt{80(20-t)}} \quad \text{and}$$

$${}_tq_x'^{(2)} = F_{T_2}(t) = 1 - \left(\frac{20-t}{20}\right)^{0.5} \quad \text{for } 0 \le t < 20$$

Solution 7.4

$$q_x'^{(1)} = \frac{1}{20} = 0.05000 \quad , \quad q_x'^{(2)} = 1 - \left(\frac{20-1}{20}\right)^{0.5} = 0.02532$$

Solution 7.5

The joint density function is not a probability, but $f_{T,J}(t,j)\,dt$ is approximately the probability that life (x) departs the group between times t and $t+dt$ as a result of cause j.

Solution 7.6

The waiting time variables $T_1, \ldots, T_r$ are assumed to be independent. Since T is the minimum of these waiting times, the event $T > t$ is the intersection of the independent events $T_i > t$. As a result, we have:

$${}_tp_x^{(\tau)} = \Pr(T > t) = \Pr(T_1 > t) \cdots \Pr(T_r > t) = {}_tp_x'^{(1)} \cdots {}_tp_x'^{(r)}$$

So for the pair of forces in Question 7.3, we have:

$${}_tp_x^{(\tau)} = \left(\frac{20-t}{20}\right) \times \left(\frac{20-t}{20}\right)^{0.5} = \left(\frac{20-t}{20}\right)^{1.5} \quad \text{for } 0 \le t \le 20$$

Solution 7.7

$$f_{T,J}(t,j) = {}_tp_x^{(\tau)}\,\mu^{(j)}(x+t) \Rightarrow$$

$$f_{T,J}(t,1) = {}_tp_x^{(\tau)}\,\mu^{(1)}(x+t) = \left(\frac{20-t}{20}\right)^{1.5} \frac{1}{20-t} = \frac{(20-t)^{0.5}}{20^{1.5}} \quad \text{for } 0 \le t \le 20$$

$$f_{T,J}(t,2) = {}_tp_x^{(\tau)}\,\mu^{(2)}(x+t) = \left(\frac{20-t}{20}\right)^{1.5} \frac{0.5}{20-t} = \frac{(20-t)^{0.5}}{2 \times 20^{1.5}} \quad \text{for } 0 \le t \le 20$$

Solution 7.8

$$\int_0^{20} f_{T,J}(t,1)\,dt + \int_0^{20} f_{T,J}(t,2)\,dt = \int_0^{20} \frac{1.5(20-t)^{0.5}}{20^{1.5}}\,dt = -\left.\frac{(20-t)^{1.5}}{20^{1.5}}\right|_0^{20} = 0 + 1 = 1$$

Solution 7.9

$$q_x^{(1)} = \Pr(T \le 1\ ,\ J=1) = \int_0^1 f_{T,J}(t,1)\,dt$$

$$= \int_0^1 \frac{(20-t)^{0.5}}{20^{1.5}}\,dt = -\left.\frac{(20-t)^{1.5}}{1.5\times 20^{1.5}}\right|_0^1 = \frac{1}{1.5}\left(1-\left(\frac{19}{20}\right)^{1.5}\right) = 0.04937$$

$$q_x^{(2)} = \Pr(T \le 1\ ,\ J=2) = \int_0^1 f_{T,J}(t,2)\,dt$$

$$= \int_0^1 \frac{0.5(20-t)^{0.5}}{20^{1.5}}\,dt = -\left.\frac{0.5(20-t)^{1.5}}{1.5\times 20^{1.5}}\right|_0^1 = \frac{0.5}{1.5}\left(1-\left(\frac{19}{20}\right)^{1.5}\right) = 0.02468$$

$$q_x^{(1)} + q_x^{(2)} = 0.04937 + 0.02468 = 0.07405$$

$$q_x'^{(1)} + q_x'^{(2)} - q_x'^{(1)}\,q_x'^{(2)} = 0.05000 + 0.02532 - 0.05000\times 0.02532 = 0.07405$$

Solution 7.10

$$\mathring{e}_x^{(\tau)} = \int_0^\infty {}_tp_x^{(\tau)}\,dt = \int_0^{20} \left(\frac{20-t}{20}\right)^{1.5} dt = -\left.\frac{(20-t)^{2.5}}{2.5\times 20^{1.5}}\right|_0^{20} = -0 + \frac{20}{2.5} = 8$$

Solution 7.11

$$\Pr(J=1) = \int_0^{20} f_{T,J}(t,1)\,dt = \int_0^{20} \frac{(20-t)^{0.5}}{20^{1.5}}\,dt = -\left.\frac{(20-t)^{1.5}}{1.5\times 20^{1.5}}\right|_0^{20} = -0 + \frac{1}{1.5} = \frac{2}{3}$$

$$\Pr(J=2) = \int_0^{20} f_{T,J}(t,2)\,dt = \int_0^{20} \frac{0.5(20-t)^{0.5}}{20^{1.5}}\,dt = -\left.\frac{0.5(20-t)^{1.5}}{1.5\times 20^{1.5}}\right|_0^{20} = -0 + \frac{0.5}{1.5} = \frac{1}{3}$$

We could have also done this without integration:

$$\Pr(J=1 \mid T=t) = \frac{\mu^{(1)}(x+t)}{\mu^{(\tau)}(x+t)} = \frac{1/(20-t)}{1.5/(20-t)} = \frac{2}{3} \quad \text{for } 0 \le t < 20$$

$$\Rightarrow T \text{ and } J \text{ are independent } \Rightarrow \Pr(J=1) = \Pr(J=1 \mid T=t) = \frac{2}{3}$$

Solution 7.12

From the table we have:

$$q_x^{(1)} = \frac{4}{100} = 0.04 \quad , \quad q_x^{(2)} = \frac{6}{100} = 0.06$$

The SUDD relations are:

$$0.04 = q_x^{(1)} = q_x'^{(1)}\left(1 - 0.5\,q_x'^{(2)}\right)$$
$$0.06 = q_x^{(2)} = q_x'^{(2)}\left(1 - 0.5\,q_x'^{(1)}\right)$$

Let's try the iterative approach:

$$q_x'^{(1)} = \frac{0.04}{1 - 0.5\,q_x'^{(2)}} \quad , \quad q_x'^{(2)} = \frac{0.06}{1 - 0.5\,q_x'^{(1)}}$$
$$q_x'^{(2)} = 0.06 \;\Rightarrow\; q_x'^{(1)} = 0.041237 \;\Rightarrow\; q_x'^{(2)} = 0.061263 \;\Rightarrow\; q_x'^{(1)} = 0.041264$$
$$\Rightarrow\; q_x'^{(2)} = 0.061264 \;\Rightarrow\; q_x'^{(1)} = 0.041264 \;\Rightarrow\; q_x'^{(2)} = 0.061264$$

The 6-place stable results are: $q_x'^{(1)} = 0.041264 \quad , \quad q_x'^{(2)} = 0.061264$

Solution 7.13

From the table we have:

$$q_x^{(1)} = \frac{4}{100} = 0.04 \quad , \quad q_x^{(1)} = \frac{6}{100} = 0.06 \quad , \quad q_x^{(\tau)} = 0.10 \quad , \quad p_x^{(\tau)} = 0.90$$

The MUDD model results in:

$$q_x'^{(1)} = 1 - p_x'^{(1)} = 1 - \left(p_x^{(\tau)}\right)^{q_x^{(1)}/q_x^{(\tau)}} = 1 - (0.90)^{0.04/0.10} = 0.041268$$
$$q_x'^{(2)} = 1 - p_x'^{(2)} = 1 - \left(p_x^{(\tau)}\right)^{q_x^{(2)}/q_x^{(\tau)}} = 1 - (0.90)^{0.06/0.10} = 0.061260$$

Solution 7.14

$$\text{SUDD} \;\Rightarrow\; {}_tp_x^{(\tau)} = {}_tp_x'^{(1)}\; {}_tp_x'^{(2)} = (1 - 0.041264t)(1 - 0.061264t)$$
$$= 1 - 0.102528\,t + 0.002528\,t^2$$
$$\text{MUDD} \;\Rightarrow\; {}_tp_x^{(\tau)} = 1 - t\,q_x^{(\tau)} = 1 - 0.10\,t$$

Solution 7.15

Under the MUDD model we saw that:

$$f_{T,J}(t,j) = {}_{k|}q_x^{(j)} \quad \text{where} \quad k=[t]$$

$$\Rightarrow \; f_{T,J}(0.5,1) = {}_{0|}q_x^{(1)} = \frac{d_x^{(1)}}{l_x^{(\tau)}} = \frac{4}{100}$$

Solution 7.16

We have the following generally valid relation:

$$0.04 = q_x^{(1)} = \int {}_t p_x'^{(2)} \; {}_t p_x'^{(1)} \; \mu^{(1)}(x+t)\, dt$$

Since decrement (1) follows the SUDD model, we have:

$${}_t q_x'^{(1)} = t\, q_x'^{(1)} \quad \Rightarrow \quad {}_t p_x'^{(1)} \mu^{(1)}(x+t) = q_x'^{(1)} \quad \text{for } 0 \le t \le 1$$

For decrement (2) we have:

$${}_t p_x'^{(2)} = \left(p_x'^{(2)}\right)^t$$

Substituting these results into the first equation results in the relation:

$$0.04 = q_x^{(1)} = \int_0^1 {}_t p_x'^{(2)} \; {}_t p_x'^{(1)} \; \mu^{(1)}(x+t)\, dt = \int_0^1 q_x'^{(1)} \left(p_x'^{(2)}\right)^t dt$$

$$= q_x'^{(1)} \left(-\frac{1-p_x'^{(2)}}{\ln\left(p_x'^{(2)}\right)} \right) = -\frac{q_x'^{(1)}\, q_x'^{(2)}}{\ln\left(1 - q_x'^{(2)}\right)}$$

Solution 7.17

By the discrete time method, we have:

$$0.06 = q_x^{(2)} = \sum_{\substack{0 \le t_k \le 1 \\ \Pr(T_2 = t_k) \ne 0}} \left(\prod_{j \ne 2} {}_{t_k} p_x'^{(j)} \right) \Pr(T_2 = t_k) = \left(1 - 0.5 q_x'^{(1)}\right) \Pr(T_2 = 0.5) = \left(1 - 0.5 q_x'^{(1)}\right) q_x'^{(2)}$$

We also have the generally valid relation:

$$0.10 = q_x^{(1)} + q_x^{(2)} = q_x'^{(1)} + q_x'^{(2)} - q_x'^{(1)} q_x'^{(2)}$$

If you subtract the first relation from the generally valid relation, then you will have 2 relations that are identical to the relations that result if both decrements follow the SUDD model. So the result will be the same as in Question 7.12.

Solution 7.18

The random present value variable for the benefit is:

$$Z = \begin{cases} 1{,}000e^{-0.05T} & J=1 \\ 2{,}000e^{-0.05T} & J=2 \end{cases}$$

From the force formulas we can calculate the joint pdf $f_{T,J}(t,j)$:

$$\mu_{30}^{(1)}(t) = \frac{1}{60-t} \quad \text{for } 0 \le t < 60 \;\Rightarrow\; {}_tp_x^{\prime(1)} = \frac{60-t}{60} \quad \text{for } 0 \le t < 60 \text{ (zero otherwise)}$$

$$\mu_{30}^{(2)}(t) = 0.01 \quad \text{for } t > 0 \;\Rightarrow\; {}_tp_x^{\prime(2)} = e^{-0.01t}$$

$$f_{T,J}(t,1) = {}_tp_x^{(\tau)}\mu^{(1)}(x+t) = \left(\frac{60-t}{60}\right)e^{-0.01t} \times \frac{1}{60-t} = \frac{e^{-0.01t}}{60} \quad \text{for } 0 \le t < 60$$

$$f_{T,J}(t,2) = {}_tp_x^{(\tau)}\mu^{(2)}(x+t) = \left(\frac{60-t}{60}\right)e^{-0.01t} \times 0.01 = \frac{(60-t)e^{-0.01t}}{6{,}000} \quad \text{for } 0 \le t < 60$$

So the single benefit premium is:

$$\begin{aligned} E[Z] &= \sum_{j=1}^{2}\left(\int_0^\infty z_{t,j}\, f_{T,J}(t,j)\,dt\right) \\ &= \int_0^\infty 1{,}000e^{-0.05t} f_{T,J}(t,1)\,dt + \int_0^\infty 2{,}000e^{-0.05t} f_{T,J}(t,2)\,dt \\ &= \int_0^{60}\left(1{,}000\,e^{-0.05t}\right)\left(\frac{e^{-0.01t}}{60}\right)dt + \int_0^{60}\left(2{,}000\,e^{-0.05t}\right)\left(\frac{(60-t)e^{-0.01t}}{6{,}000}\right)dt \end{aligned}$$

Solution 7.19

$$\begin{aligned} \text{APV} &= \frac{1{,}000\,q_x^{(1)}}{1.05} + \frac{1{,}000\;{}_{1|}q_x^{(1)}}{1.05^2} + \frac{2{,}000\,q_x^{(2)}}{1.05} + \frac{2{,}000\;{}_{1|}q_x^{(2)}}{1.05^2} \\ &= \frac{1{,}000(0.035)}{1.05} + \frac{1{,}000(0.045)}{1.05^2} + \frac{2{,}000(0.001)}{1.05} + \frac{2{,}000(0.002)}{1.05^2} \\ &= 79.68254 \end{aligned}$$

Solution 7.20

Since the premiums are paid at time 0 and at time 1 (if surviving), the APV of premium is equal to:

$$P + P\,v\,p_x^{(\tau)} = P\left(1 + \frac{964/1{,}000}{1.05}\right) = 1.91810P$$

As a result of the equivalence principle, we have:

$$1.91810P = \text{APV of Premium} = \text{APV of Benefit} = 79.68254$$
$$\Rightarrow\ P = 41.54254$$

Solution 7.21

All members of the pension plan must retire at age 65 at the latest. However, in this plan the employer also allows some members to retire earlier. Benefits are based on the total length of time in employment, which includes the ten years that this member has worked prior to age 63. So, if the plan member retires at age 64 years and 6 months he will receive a benefit based on his 11.5 years in employment. Hence, the actuarial present value of the normal retirement benefits is given by:

$$40{,}000\left(\frac{10.5}{100}\times\frac{d_{63}^{(1)}}{l_{63}^{(\tau)}}\times\frac{(1.06)^{0.5}}{(1.07)^{0.5}}a_{63.5}^{(1)}+\frac{11.5}{100}\times\frac{d_{64}^{(1)}}{l_{63}^{(\tau)}}\times\frac{(1.06)^{1.5}}{(1.07)^{1.5}}a_{64.5}^{(1)}+\frac{12}{100}\times\frac{d_{65}^{(1)}}{l_{63}^{(\tau)}}\times\frac{(1.06)^{2}}{(1.07)^{2}}a_{65}^{(1)}\right)$$

$$=40{,}000\left(\frac{10.5}{100}\times\frac{110}{1{,}000}\times\frac{(1.06)^{0.5}}{(1.07)^{0.5}}\times16.3+\frac{11.5}{100}\times\frac{145}{1{,}000}\times\frac{(1.06)^{1.5}}{(1.07)^{1.5}}\times15.9+\frac{12}{100}\times\frac{476}{1{,}000}\times\frac{(1.06)^{2}}{(1.07)^{2}}\times14.9\right)$$

$$=\$51{,}362.47$$

This pension plan offers higher benefits to members who are forced to retire through ill health. The benefits are based on the number of years that would have been worked if the member had remained healthy until age 65. This would be a particularly valuable benefit for someone who became disabled through an accident at an early age for example. However, an illness that forces a worker to retire may also have an impact on their life expectancy, so we value the annuity using a higher rate of mortality. Hence, the actuarial present value of the ill-health retirement benefits is given by:

$$40{,}000\left(\frac{12}{100}\times\frac{d_{63}^{(2)}}{l_{63}^{(\tau)}}\times\frac{(1.06)^{0.5}}{(1.07)^{0.5}}a_{63.5}^{(2)}+\frac{12}{100}\times\frac{d_{64}^{(2)}}{l_{63}^{(\tau)}}\times\frac{(1.06)^{1.5}}{(1.07)^{1.5}}a_{64.5}^{(2)}\right)$$

$$=40{,}000\left(\frac{12}{100}\times\frac{70}{1{,}000}\times\frac{(1.06)^{0.5}}{(1.07)^{0.5}}\times12.7+\frac{12}{100}\times\frac{95}{1{,}000}\times\frac{(1.06)^{1.5}}{(1.07)^{1.5}}\times12.3\right)$$

$$=\$9{,}777.57$$

When a member withdraws from the pension plan, perhaps to take up a job elsewhere, the plan offers a deferred pension payable from age 65. Benefits are based on the salary and years of employment at the date of withdrawal. So, the actuarial present value of the withdrawal benefits is given by:

$$40,000\left(\frac{10.5}{100}\times\frac{d_{63}^{(3)}}{l_{63}^{(\tau)}}\times\frac{(1.06)^{0.5}}{(1.07)^{2}}a_{65}^{(1)}+\frac{11.5}{100}\times\frac{d_{64}^{(3)}}{l_{63}^{(\tau)}}\times\frac{(1.06)^{1.5}}{(1.07)^{2}}a_{65}^{(1)}\right)$$

$$=40,000\left(\frac{10.5}{100}\times\frac{55}{1,000}\times\frac{(1.06)^{0.5}}{(1.07)^{2}}\times 14.9+\frac{11.5}{100}\times\frac{35}{1,000}\times\frac{(1.06)^{1.5}}{(1.07)^{2}}\times 14.9\right)$$

$$=\$5,381.83$$

If the plan member dies, the pension plan pays a lump sum of 4 times his salary at the time of death. We are assuming that deaths occur half way through the year. So, if the plan member dies at age 63 years and 6 months (and has not already retired or withdrawn from the scheme) he will receive a benefit based on a salary of $40,000 plus salary increases for half a year. Hence, the actuarial present value of the death benefits is given by:

$$4\times 40,000\left(\frac{d_{63}^{(4)}}{l_{63}^{(\tau)}}\times\frac{(1.06)^{0.5}}{(1.07)^{0.5}}+\frac{d_{64}^{(4)}}{l_{63}^{(\tau)}}\times\frac{(1.06)^{1.5}}{(1.07)^{1.5}}\right)$$

$$=160,000\left(\frac{6}{1,000}\times\frac{(1.06)^{0.5}}{(1.07)^{0.5}}+\frac{8}{1,000}\times\frac{(1.06)^{1.5}}{(1.07)^{1.5}}\right)$$

$$=\$2,217.60$$

So, the total value of plan benefits for this member is $68,739.47.

