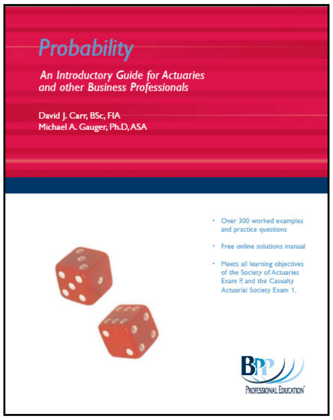

Probability

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Solutions to practice questions – Chapter 2

Solution 2.1

(i) There are $n_1 = 4$ ways to choose the color of the cap, $n_2 = 6$ ways to choose the color of the shirt, and $n_3 = 2$ ways to choose the color of the pants. So the number of different color combinations for the uniform is:

$$n_1 n_2 n_3 = 4 \times 6 \times 2 = 48$$

(ii) There are $n_1 = 4$ ways to choose the color of the cap, $n_2 = 4$ ways to choose the color of the shirt, and $n_3 = 1$ way to choose the color of the pants. So the number of different color combinations for the uniform is:

$$n_1 n_2 n_3 = 4 \times 4 \times 1 = 16$$

(iii) First, note that the set of pants colors is a subset of the set of cap colors, which in turn is a subset of the shirt colors. If we first select the pant colors, there are $n_1 = 2$ choices. Once the color of the pants is determined, there are $n_2 = 4 - 1 = 3$ ways to choose a different color for the cap. Once the color of both the pants and the cap have been chosen, there are $n_3 = 6 - 2 = 4$ ways to choose a different color for the shirt. So the number of different color combinations for the uniform is:

$$n_1 n_2 n_3 = 2 \times 3 \times 4 = 24$$

Solution 2.2

(i) The number of ordered samples of 3 objects from a set of 10 objects with replacement is:

$$10^3 = 1,000$$

(ii) The number of ordered samples of 3 objects from a set of 10 objects without replacement is:

$$_{10}P_3 = 10 \times 9 \times 8 = 720$$

(iii) This part of the problem is quite difficult. Let n_1, n_2, and n_3 be the number of possible choices for the first, second and third balls chosen. Then notice that the values of n_2 and n_3 depend on whether the previous balls were odd or even. The following table gives a breakdown of all the possible outcomes:

Ball 1	n_1	Ball 2	n_2	n_3	$n_1 n_2 n_3$
Odd	5	Odd	4	8	160
Odd	5	Even	5	9	225
Even	5	Odd	5	9	225
Even	5	Even	5	10	250

Hence the total number of possible outcomes if the odd balls are not replaced is:

$$160+225+225+250=860$$

Solution 2.3

Any runner can win at most one medal. So we are choosing an ordered sample of 3 objects from a set of 30 objects without replacement. The number of possible outcomes is:

$$_{30}P_3 = 30\times 29\times 28 = 24,360$$

Solution 2.4

The number of heads could be 3, 4 or 5. The number of possible outcomes that include three or more heads is:

$$_5C_3 + {_5C_4} + {_5C_5} = 10+5+1=16$$

Solution 2.5

(i) The seven distinct letters in WYOMING can be rearranged in $7!=5,040$ ways.

(ii) Since there are 2 A's in ARIZONA, the number of ordered arrangements is:

$$\frac{7!}{2!}=2,520$$

(iii) There are 13 letters in the word MASSACHUSETTS. The letter A occurs twice, the letter T occurs twice, and the letter S occurs four times. Each of the other five letters occurs just once. So the number of ordered rearrangements is:

$$\frac{13!}{4!2!2!}=64,864,800$$

Solution 2.6

The number of ways to select three groups of six people is:

$$\frac{18!}{6!6!6!}=17,153,136$$

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Solution 2.7

Let E denote the event of 3 or more heads. It is much easier to count the ways in which we can observe 2 or fewer heads. The required probability is:

$$\Pr(E)=1-\Pr(E')=1-\Pr(H_0)-\Pr(H_1)-\Pr(H_2)$$

where H_i denotes the event that i heads are observed before the game stops.

It is easy to see that $\Pr(H_0)=0.5^3$ since a tail must occur on 3 consecutive independent tosses of a fair coin.

One possible way that H_1 can occur is the sequence $HTTT$. The probability of this sequence is 0.5^4. Since the final toss must be a tail, there are ${}_3C_1=3$ ways to rearrange HTT, so we have:

$$\Pr(H_1)=3\times 0.5^4$$

One possible way that H_2 can occur is $HHTTT$. The probability of this sequence is 0.5^5. Since the final toss must be a tail, there are ${}_4C_2=6$ ways to rearrange $HHTT$, so we have:

$$\Pr(H_2)=6\times 0.5^5$$

Finally, we have:

$$\begin{aligned}\Pr(E)&=1-\Pr(E')=1-\Pr(H_0)-\Pr(H_1)-\Pr(H_2)\\&=1-\left(0.5^3+3\times 0.5^4+6\times 0.5^5\right)=0.50\end{aligned}$$

Solution 2.8

The order in which the funds is selected does not matter. The 10 funds can be selected from the 100 funds in ${}_{100}C_{10}$ ways. The number of ways of selecting 3 funds from the top 25 funds is ${}_{25}C_3$, the number of ways of selecting 5 funds from the middle 50 funds is ${}_{50}C_5$, and the number of ways of selecting 2 of the bottom 25 funds is ${}_{25}C_2$. So the required probability is:

$$\frac{{}_{25}C_3\times{}_{50}C_5\times{}_{25}C_2}{{}_{100}C_{10}}=\frac{(2{,}300)(2{,}118{,}760)(300)}{1.731031\times 10^{13}}=0.0845$$

Solution 2.9

(i) Each of the ${}_{48}C_6=12{,}271{,}512$ six-digit combinations is equally likely. There are ${}_9C_2=36$ ways to select two of the numbers 1-9, and there are ${}_{39}C_4=82{,}251$ ways to choose four of the numbers 10-48. So, the probability that exactly two of the winning six numbers are single digit numbers is:

$$\frac{({}_9C_2)({}_{39}C_4)}{{}_{48}C_6}=\frac{36\times 82{,}251}{12{,}271{,}512}=0.2413$$

(ii) Similarly, that probability that two of the winning numbers are single digit and the rest of the winning combination consists of one number from 10-19, one number from 20-29, one number from 30-39, and one number from 40-48 is:

$$\frac{({}_9C_2)({}_{10}C_1)({}_{10}C_1)({}_{10}C_1)({}_9C_1)}{{}_{48}C_6}=\frac{36\times 10\times 10\times 10\times 9}{12{,}271{,}512}=0.0264$$

Solution 2.10

Let *M* denote the event that the largest claim for a week is on a medical policy. Similarly, let *A* and *H* respectively denote the event that the largest claim during a week is from an automobile policy or a homeowner's policy.

The event that the largest claim is from a medical policy on at least 2 more occasions than from an automobile policy corresponds to the following (unordered) combinations:

MMHH *MMMA* *MMMH* *MMMM*

The probability of the sequence $MMHH$ is $0.6^2 \times 0.1^2$ and there are $4!/(2!2!)$ possible rearrangements of these 4 letters. Using the same idea with the other sequences listed above, we see that the probability that the largest claim is from a medical policy on at least 2 more occasions than from an automobile policy is:

$$\frac{4!}{2!2!}0.6^2 \times 0.1^2 + \frac{4!}{3!}0.6^3 \times 0.3 + \frac{4!}{3!}0.6^3 \times 0.1 + 0.6^4 = 0.4968$$

Solution 2.11

Let *N* denote the event that no accidents occur in a month, and let *A* denote the complementary event that there is at least one accident in a month. The critical time period to consider is 7 months. At the end of 7 months, we will have witnessed either at least two months with accidents, or at least six accident-free months.

The probability that the first 7 months are all accident-free is:

$$0.76^7 = 0.1465$$

The probability that there are exactly 6 accident-free months in the first 7 months is:

$${}_7C_1 \times 0.76^6 \times 0.24 = 0.3237$$

Hence the required probability is $0.1465 + 0.3237 = 0.4702$.

If you cannot see why 7 months is the critical time, it might help you to write out some possibilities for this question and see which work. Starting with no months with accidents, then we could have NNNNNN to satisfy the condition in the question. Now consider one month with accidents, for example ANNNNNN. We could have any of the seven arrangements of this possibility with the 'A' in any of the seven positions. If the 'A' is in the seventh position, then it is equivalent to NNNNNN, so we can now ignore that possibility. This means it is necessary to consider what happens in the 7th month in order to be assured of getting at least 6 accident free months. Finally consider two months with accidents. The only way this could work would be to have one A in the first 7 months and then the next A in the 8th month. This is equivalent to have ANNNNNN. This brings us back to seven months being the critical time.

Solution 2.12

From the description in the question, the event of interest is equivalent to one minor and one moderate accident in the month, or two minor accidents in the month. So the probability of this event is:

$${}_2C_1 \times \Pr(\text{minor}) \times \Pr(\text{moderate}) + \left(\Pr(\text{minor})\right)^2 = 2 \times 0.5 \times 0.4 + 0.5^2 = 0.65$$

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Solution 2.13

The unfortunate coach will be fired if the Cadavers win 0, 1, or 2 of the next 7 road games.

The probability is thus:

$$\underbrace{{}_7C_0 \times 0.65^7}_{\text{7 losses}} + \underbrace{{}_7C_1 \times 0.35^1 \times 0.65^6}_{\text{6 losses, 1 win}} + \underbrace{{}_7C_2 \times 0.35^2 \times 0.65^5}_{\text{5 losses, 2 wins}} = 0.5323$$

Solution 2.14

If you win 4 or 5 matches of the next 5, you must automatically win consecutive matches. The probability of at least 4 wins is:

$$0.4^5 + {}_5C_1 \times 0.4^4 \times 0.6^1 = 0.08704$$

There are ${}_5C_3 = 10$ different ways that you could win 3 matches. Only one of these ways would not result in consecutive wins (*ie* WLWLW). So the probability of winning 3 matches and winning consecutive matches is:

$$\left({}_5C_3 - 1\right) \times 0.4^3 \times 0.6^2 = 0.20736$$

There are exactly 4 ways that you could win two consecutive matches and yet win only 2 of the 5 matches (WWLLL, LWWLL, LLWWL, LLLWW). The probability of this outcome is:

$$4 \times 0.4^2 \times 0.6^3 = 0.13824$$

Hence, the total probability is 0.43264.

Solution 2.15

There are ${}_{59}C_2 = 1{,}711$ equally likely ways to fill the two adjacent seats with a passenger or an empty space. Let E be the event that a member of the college soccer team will occupy at least one of the two adjacent seats. It is simpler to count the number of combinations corresponding to the complementary event.

For the complementary event, occupiers are chosen from the other 44 passengers and empty seat assignments:

$$\Pr(E) = 1 - \Pr(E') = 1 - \frac{{}_{44}C_2}{1{,}711} = 1 - \frac{946}{1{,}711} = 0.4471$$

Solution 2.16

The other positions in George's group can be filled in ${}_{23}C_2 = 253$ ways. The number of ways in which these positions can be filled by students other than George's friends is ${}_{19}C_2 = 171$. Hence the required probability is:

$$\frac{{}_{19}C_2}{{}_{23}C_2} = \frac{171}{253} = 0.6759$$

Solution 2.17

Consider the following table where n is the number of George's friends. The probability that George has at least one friend in his group if he has n friends in the class is:

$$\Pr(\text{At least 1 friend in group}) = 1 - \frac{_{23-n}C_2}{_{23}C_2} = 1 - \frac{(23-n)(22-n)}{23\times 22} = \frac{45n-n^2}{506}$$

From the following table, it is clear that he must have 7 friends for there to be at least a 50% chance that he has at least one friend in his group.

n	4	5	6	7
Pr(At least 1 friend in group)	0.324	0.395	0.462	0.526

Solution 2.18

There are $_{49}P_2 = 49\times 48 = 2,352$ ways to choose the fourth and fifth cards in his hand. Once again it is simpler to compute the probability of the complementary event, namely, he will not end up with a pair.

The number of ways to choose a fourth card that is neither a jack, nor a two, nor a nine is $_{40}C_1 = 40$.

Once this card has been chosen, we must choose a fifth card that does not match any of the other four cards, and there are $_{36}C_1 = 36$ ways to do so.

So the probability that he will end up with at least one pair is:

$$1 - \frac{40\times 36}{2,352} = 1 - \frac{1,440}{2,352} = 0.3878$$

Solution 2.19

The number of ways to match exactly 3 of the 6 winning numbers is $(_6C_3)(_{43}C_3) = 246,820$. In a similar fashion we can count the number of ways of matching exactly 4, 5, or 6 of the winning numbers. The probability of matching at least 3 of the 6 winning numbers is:

$$\frac{(_6C_3)(_{43}C_3)+(_6C_4)(_{43}C_2)+(_6C_5)(_{43}C_1)+(_6C_6)(_{43}C_0)}{_{49}C_6} = \frac{260,624}{13,983,816} = 0.0186$$

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Solution 2.20

Let H denote the event that a new driver is high risk, let M denote a moderate risk driver, and let L denote a low risk driver. Then we have:

$$\Pr(H)=0.2\ ,\ \Pr(M)=0.3\ ,\ \Pr(L)=0.5$$

For there to be at least 2 more high risk drivers than low risk drivers among the next 4 new drivers we could have any of the following:

$$2H, 2M, 0L \qquad \Pr = \frac{4!}{2!2!0!} \times 0.2^2 \times 0.3^2 = 0.0216$$

$$3H, 1M, 0L \qquad \Pr = \frac{4!}{3!1!0!} \times 0.2^3 \times 0.3^1 = 0.0096$$

$$3H, 0M, 1L \qquad \Pr = \frac{4!}{3!0!1!} \times 0.2^3 \times 0.5^1 = 0.0160$$

$$4H, 0M, 0L \qquad \Pr = \frac{4!}{4!0!0!} \times 0.2^4 = 0.0016$$

Hence the total probability is:

$$0.0216 + 0.0096 + 0.0160 + 0.0016 = 0.0488$$