Probability, Fourth Edition

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Solutions to practice questions – Chapter 5

Solution 5.1

Let X be the number of visits (out of the next 10 visits) that result in a referral. Then X follows a binomial distribution with $n=10$ and $p=0.28$, and the required probability is:

$$\Pr(X=4)=\left({}_{10}C_4\right)\left(0.28^4\right)\left(0.72^6\right)=0.1798$$

Solution 5.2

Let X be the number of people who suffer a side effect. Then X follows a binomial distribution with $n=1{,}000$ and $p=0.005$, and the required probability is:

$$\begin{aligned}\Pr(X\le 1)&=\Pr(X=0)+\Pr(X=1)\\&=0.995^{1{,}000}+\left({}_{1{,}000}C_1\right)\left(0.005^1\right)\left(0.995^{999}\right)\\&=0.0067+0.0334=0.0401\end{aligned}$$

Solution 5.3

The probability that a report is filed is:

$$\Pr(\text{More than one injury})=0.23+0.17+0.09+0.04=0.53$$

Let X be the number of reports filed for the next 20 games. Then X follows a binomial distribution with $n=20$ and $p=0.53$.

The expected number of reports is:

$$E[X]=np=20\times 0.53=10.6$$

The standard deviation of the number of reports is:

$$\sqrt{npq}=\sqrt{20\times 0.53\times 0.47}=2.232$$

Solution 5.4

Using the moment generating function:

$$M_X(t)=\left(pe^t+q\right)^n$$

$$\Rightarrow M'_X(t)=(n)(pe^t)(pe^t+q)^{n-1}$$

$$\Rightarrow M''_X(t)=(n)(pe^t)(pe^t+q)^{n-1}+(n)(n-1)(pe^t)^2(pe^t+q)^{n-2} \qquad \text{(product rule)}$$

(i) The mean is:

$$E[X]=M'_X(0)=(n)(p)(p+q)^{n-1}=np \qquad \text{since } p+q=1$$

(ii) The variance is:

$$\text{var}(X)=E\left[X^2\right]-\left(E[X]\right)^2$$

$$E\left[X^2\right]=M''_X(0)=(n)(p)(p+q)^{n-1}+(n)(n-1)(p)^2(p+q)^{n-2}=np+n(n-1)p^2$$

$$\Rightarrow \quad \text{var}(X)=np+n(n-1)p^2-\left(np\right)^2=np-np^2=np(1-p)=npq$$

Solution 5.5

The number of claims, X, follows the binomial distribution with $n=10$ and $p=0.125$.

The expected number of claims is:

$$E\left[X\right]=np=10\times 0.125=1.25$$

The variance is:

$$\text{var}\left(X\right)=npq=10\times 0.125\times 0.875=1.09375$$

The standard deviation is:

$$\sqrt{\text{var}\left(X\right)}=\sqrt{1.09375}=1.04583$$

The reserve is:

$$10\times\left(1.25+2\times 1.04583\right)=33.417 \text{ (\$million)}$$

Solution 5.6

The number of hurricanes, X, can be modeled using a binomial distribution if we treat each year as a Bernoulli trial with $\Pr\left(\text{Success}\right)=\Pr\left(\text{Hurricane}\right)=p=0.05$ and the number of trials (years) is $n=20$.

Using the binomial distribution probability function:

$$\Pr\left(X<3\right)=\Pr\left(X=0\right)+\Pr\left(X=1\right)+\Pr\left(X=2\right)$$

$$=\left({}_{20}C_0\right)\left(0.05^0\right)\left(0.95^{20}\right)+\left({}_{20}C_1\right)\left(0.05^1\right)\left(0.95^{19}\right)+\left({}_{20}C_2\right)\left(0.05^2\right)\left(0.95^{18}\right)$$

$$=0.35849+0.37735+0.18868=0.9245$$

Solution 5.7

Let's define X as the number of tourists who do show up. Then X has a binomial distribution with $n = 21$ trials and with $p = 0.98$ (we define "success" as a tourist showing up).

The tour operator's revenue, R, is:

- $21 \times 50 = 1{,}050$ if $X \leq 20$ (a seat is available for everyone)
- $21 \times 50 - 100 = 950$ if all 21 tourists show up

The expected revenue is thus:

$$
\begin{aligned}
E[R] &= 1{,}050 \times \Pr(X \leq 20) + 950 \times \Pr(X = 21) \\
&= 1{,}050 \times (1 - \Pr(X = 21)) + 950 \times \Pr(X = 21) \\
&= 1{,}050 \times \left(1 - 0.98^{21}\right) + 950 \times 0.98^{21} = 985
\end{aligned}
$$

Solution 5.8

Let X be the annual number of claims.

The probability of no more than 2 claims being filed in the next year is:

$$
\begin{aligned}
\Pr(X \leq 2) &= \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) \\
&= p^r + \frac{r!}{1!(r-1)!} p^r q + \frac{(r+1)!}{2!(r-1)!} p^r q^2 \\
&= 0.7^8 + \frac{8!}{1!7!} \times 0.7^8 \times 0.3 + \frac{9!}{2!7!} \times 0.7^8 \times 0.3^2 \\
&= 0.0576 + 0.1384 + 0.1868 = 0.3828
\end{aligned}
$$

Solution 5.9

Let X be the annual number of claims.

We must solve two simultaneous equations involving the two parameters:

$$3 = E[X] = \frac{rq}{p} \qquad 7.5 = \text{var}(X) = \frac{rq}{p^2}$$

If we divide the first equation by the second, we see that:

$$p = \frac{E[X]}{\text{var}(X)} = \frac{3}{7.5} = 0.4 \quad \Rightarrow \quad q = 1 - p = 0.6 \quad \Rightarrow \quad r = 2$$

Hence the required probability is:

$$\Pr(X = 3) = \frac{(r+2)!}{3!(r-1)!} p^r q^3 = \frac{4!}{3!1!} \times 0.4^2 \times 0.6^3 = 0.13824$$

Solution 5.10

Using the moment generating function:

$$M_X(t)=\left(\frac{1-qe^t}{p}\right)^{-r}=p^r\left(1-qe^t\right)^{-r}$$

$$\Rightarrow M'_X(t)=(p^r)(-qe^t)(-r)(1-qe^t)^{-r-1}=(rqp^r)(e^t)(1-qe^t)^{-r-1}$$

$$\Rightarrow M''_X(t)=(rqp^r)\left[(e^t)(1-qe^t)^{-r-1}+(e^t)(-qe^t)(-r-1)(1-qe^t)^{-r-2}\right] \quad \text{(product rule)}$$

(i) The mean is:

$$E[X]=M'_X(0)=(rqp^r)(1)(1-q)^{-r-1}=(rqp^r)(p)^{-(r+1)}=\frac{rq}{p}$$

(ii) The variance is:

$$\text{var}(X)=E\left[X^2\right]-(E[X])^2$$

$$E\left[X^2\right]=M''_X(0)=(rqp^r)\left[(1-q)^{-(r+1)}+(q)(r+1)(1-q)^{-(r+2)}\right]=\frac{rq}{p}+\frac{r(r+1)q^2}{p^2}$$

$$\Rightarrow \quad \text{var}(X)=\frac{rq}{p}+\frac{r(r+1)q^2}{p^2}-\left(\frac{rq}{p}\right)^2=\frac{rq}{p}+\frac{rq^2}{p^2}=\frac{rq(p+q)}{p^2}=\frac{rq}{p^2}$$

Solution 5.11

Let X be the number of accident-free months before the third month in which at least one accident occurs.

Then X follows a negative binomial distribution with $p=0.3$ and $r=3$, and the required probability is:

$$\Pr(X\geq 3)=1-\Pr(X=0)-\Pr(X=1)-\Pr(X=2)$$

$$=1 - p^r - \frac{r!}{1!(r-1)!}p^r q - \frac{(r+1)!}{2!(r-1)!}p^r q^2$$

$$=1 - 0.3^3 - \frac{3!}{1!2!}\times 0.3^3\times 0.7 - \frac{4!}{2!2!}\times 0.3^3\times 0.7^2$$

$$=1-0.027-0.0567-0.07938=0.83692$$

Solution 5.12

The probability of success (*ie* all 5 coins are heads) on a single trial is:

$$\left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

Then N (the number of trials until all five coins are heads) follows a geometric distribution (as detailed on page 117) with $p = \frac{1}{32}$ and $q = 1 - p = \frac{31}{32}$. Hence:

$$\Pr(N \geq 20) = 1 - \Pr(N \leq 19) = 1 - \left(1 - q^{19}\right) = q^{19} = \left(\frac{31}{32}\right)^{19} = 0.5470$$

Solution 5.13

The probability that there are at least 4 injuries in a game is $0.09 + 0.04 = 0.13$.

Let X be the number of games before the first game in which there are at least 4 injuries. Then X follows a geometric distribution with $p = 0.13$ and $q = 1 - p = 0.87$. Hence:

$$E[X] = \frac{q}{p} = \frac{0.87}{0.13} = 6.6923$$

Solution 5.14

Let X be the random number of trucks with engine problems in this sample. If we define a truck with engine problems to be Type 1 and a truck without engine problems to be Type 2, then X follows a hypergeometric distribution with parameters:

$$m = 20 \qquad m_1 = 6 \qquad m_2 = 14 \qquad n = 4$$

The probability that exactly 2 of the 4 trucks tested have engine problems is:

$$\Pr(X = 2) = \frac{\left({}_6C_2\right)\left({}_{14}C_2\right)}{{}_{20}C_4} = 0.2817$$

Solution 5.15

Let X be the random number of hearts in the sample. If we define a heart to be Type 1 and all other suits to be Type 2, then X follows a hypergeometric distribution with parameters:

$$m = 52 \qquad m_1 = 13 \qquad m_2 = 39 \qquad n = 5$$

Then we have:

(i) $E[X] = \frac{nm_1}{m} = \frac{5 \times 13}{52} = 1.25$

(ii) $\operatorname{var}(X) = n\left(\frac{m_1}{m}\right)\left(\frac{m_2}{m}\right)\left(\frac{m-n}{m-1}\right) = 5 \times \frac{13}{52} \times \frac{39}{52} \times \frac{47}{51} = 0.8640$

(iii) $\Pr(X = 0) = \frac{\left({}_{13}C_0\right)\left({}_{39}C_5\right)}{{}_{52}C_5} = 0.2215$

Solution 5.16

Let X be the number of accidents in the next year. Then X follows a Poisson distribution with $\lambda = 5$, and the required probability is:

$$\Pr(X=3) = \frac{e^{-\lambda}\lambda^3}{3!} = \frac{e^{-5}5^3}{3!} = 0.1404$$

Solution 5.17

Let X be the number of hurricanes in a particular year. Then X follows a Poisson distribution with $\lambda = 2.8$, and we have:

$$\Pr(X=0) = e^{-\lambda} = e^{-2.8} = 0.06081$$

$$\begin{aligned}\Pr(X \geq 3) &= 1 - \Pr(X=0) - \Pr(X=1) - \Pr(X=2) \\ &= 1 - e^{-2.8} - \frac{e^{-2.8}2.8}{1!} - \frac{e^{-2.8}2.8^2}{2!} \\ &= 1 - 0.06081 - 0.17027 - 0.23838 = 0.53054\end{aligned}$$

Hence, the required probability is:

$$\Pr\left(X \geq 3 \middle| X \geq 1\right) = \frac{\Pr(X \geq 3 \cap X \geq 1)}{\Pr(X \geq 1)} = \frac{\Pr(X \geq 3)}{1 - \Pr(X=0)} = \frac{0.53054}{1 - 0.06081} = 0.5649$$

Solution 5.18

For a Poisson distribution we have:

$$\Pr(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

Hence:

$$3 = \frac{\Pr(X=2)}{\Pr(X=4)} = \frac{e^{-\lambda}\lambda^2/2!}{e^{-\lambda}\lambda^4/4!} = \frac{12}{\lambda^2} \quad \Rightarrow \quad \lambda = 2$$

$$\Rightarrow \quad \text{var}(X) = \lambda = 2$$

Solution 5.19

Let N be the number of consecutive days of rain beginning April 1.

Then N follows a Poisson distribution with $\lambda = 0.6$, so:

$$\Pr(N = n) = \frac{e^{-0.6} 0.6^n}{n!} \quad \text{for } n = 0, 1, \cdots$$

The amount paid, X, is zero with probability:

$$\Pr(N = 0) = \frac{e^{-0.6} 0.6^0}{0!} = 0.54881$$

The amount paid, X, is 1,000 with probability:

$$\Pr(N = 1) = \frac{e^{-0.6} 0.6^1}{1!} = 0.32929$$

The amount paid, X, is 2,000 with probability:

$$\Pr(N \geq 2) = 1 - \Pr(N = 0) - \Pr(N = 1) = 0.12190$$

Thus, we have:

$$E[X] = 0 \times 0.54881 + 1,000 \times 0.32929 + 2,000 \times 0.1219 = 573.1$$

$$E\left[X^2\right] = 0^2 \times 0.5488 + 1,000^2 \times 0.3293 + 2,000^2 \times 0.1219 = 816,900$$

$$\text{var}(X) = E\left[X^2\right] - \left(E[X]\right)^2 = 488,456$$

So the standard deviation is $\sqrt{\text{var}(X)} = 699$.

Solution 5.20

The sample mean is:

$$\bar{X} = \frac{X_1 + X_2 + X_3 + X_4}{4}$$

Let's also define the sample total:

$$S = X_1 + X_2 + X_3 + X_4 = 4\bar{X}$$

By the additive property, S follows a Poisson distribution with parameter 4λ.

Hence the required probability is:

$$\begin{aligned} \Pr\left(\bar{X} < 0.5\right) &= \Pr(S < 2) \\ &= \Pr(S = 0) + \Pr(S = 1) \\ &= e^{-4\lambda} + 4\lambda e^{-4\lambda} \\ &= e^{-4\lambda}(1 + 4\lambda) \end{aligned}$$

