



Probability, Third Edition

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Solutions to practice questions – Chapter 6

Solution 6.1

Let the losses due to storm, fire, and theft be denoted X_1 , X_2 and X_3 respectively.

$$\begin{aligned} \Pr(\max\{X_1, X_2, X_3\} > 2.5) &= 1 - \Pr(\max\{X_1, X_2, X_3\} \leq 2.5) \\ &= 1 - \Pr(X_1 \leq 2.5) \times \Pr(X_2 \leq 2.5) \times \Pr(X_3 \leq 2.5) \end{aligned}$$

The distribution function for a uniform distribution with range $[a, b]$ is:

$$F(x) = \frac{x - a}{b - a}$$

So, we can calculate the probability as:

$$\begin{aligned} \Pr(\max\{X_1, X_2, X_3\} > 2.5) &= 1 - \Pr(X_1 \leq 2.5) \times \Pr(X_2 \leq 2.5) \times \Pr(X_3 \leq 2.5) \\ &= 1 - (1) \left(\frac{2.5}{3}\right) \left(\frac{2.5}{4}\right) = 1 - 0.521 = 0.479 \end{aligned}$$

Solution 6.2

We have:

$$\begin{aligned} E[X] &= \frac{a+b}{2} = 1,879 \\ \text{var}(X) &= \frac{(b-a)^2}{12} = 507 \Rightarrow b-a = \sqrt{12 \times 507} = 78 \end{aligned}$$

Hence:

$$\begin{aligned} a+b &= 2 \times 1,879 = 3,758 \\ \Rightarrow a+(78+a) &= 3,758 \\ \Rightarrow a &= 1,840 \\ \Rightarrow b &= 1,918 \end{aligned}$$

Solution 6.3

⇒ From 2010, this question is no longer covered by the Exam P syllabus.

Solution 6.4

We need:

$$E[X] = \int_0^{\infty} x 3 \times 2,000^3 (x + 2,000)^{-4} dx$$

Integrating by parts:

$$\begin{aligned} E[X] &= -\frac{x 2,000^3}{(x + 2,000)^3} \Bigg|_0^{\infty} + \int_0^{\infty} 2,000^3 (x + 2,000)^{-3} dx \\ &= 0 - \frac{2,000^3}{2(x + 2,000)^2} \Bigg|_0^{\infty} \\ &= 1,000 \end{aligned}$$

Solution 6.5

We want to calculate the following:

$$\Pr(X < 2 | X > 1.5) = \frac{\Pr(X < 2 \cap X > 1.5)}{\Pr(X > 1.5)} = \frac{\Pr(1.5 < X < 2)}{\Pr(X > 1.5)}$$

Now:

$$\Pr(X > 1.5) = \int_{1.5}^{\infty} 3x^{-4} dx = -\frac{1}{x^3} \Bigg|_{1.5}^{\infty} = \frac{1}{1.5^3} = 0.296296$$

$$\Pr(1.5 < X < 2) = \int_{1.5}^2 3x^{-4} dx = -\frac{1}{x^3} \Bigg|_{1.5}^2 = \frac{1}{1.5^3} - \frac{1}{2^3} = 0.171296$$

So:

$$\Pr(X < 2 | X > 1.5) = \frac{0.171296}{0.296296} = 0.578125 \quad (\text{exactly})$$

Solution 6.6

First we need the value of c . This is found from:

$$\int_3^{\infty} c x^{-4} dx = 1$$

$$\Rightarrow \frac{1}{c} = -\frac{1}{3x^3} \Big|_3^{\infty} = \frac{1}{81}$$

$$\Rightarrow c = 81$$

The expected lifetime is then:

$$E[X] = \int_3^{\infty} x 81 x^{-4} dx = -\frac{81}{2x^2} \Big|_3^{\infty} = 4.5$$

Solution 6.7

For the exponential distribution with mean 50, we have:

$$\Pr(X > x) = 1 - F(x) = e^{-x/50}$$

Hence:

$$\Pr(X > 200 | X > 50) = \frac{\Pr(X > 200)}{\Pr(X > 50)} = \frac{e^{-200/50}}{e^{-50/50}} = \frac{e^{-4}}{e^{-1}} = e^{-3} = 0.0498$$

Alternatively, using the memoryless property we have:

$$\Pr(X > 200 | X > 50) = \Pr(X > 150) = e^{-150/50} = e^{-3} = 0.0498$$

Solution 6.8

Let X be the random time (in days) until the high-risk driver has an accident. Since X follows an exponential distribution, the cdf is:

$$F(x) = 1 - e^{-x/\theta}$$

We are given that

$$0.30 = \Pr(X < 50) = 1 - e^{-50/\theta}$$

$$\Rightarrow e^{-50/\theta} = 0.70$$

Hence:

$$\Pr(X < 80) = 1 - e^{-80/\theta} = 1 - \left(e^{-50/\theta}\right)^{8/5} = 1 - 0.70^{8/5} = 0.4349$$

Note: It is unnecessary to calculate the precise value of θ . If you did solve for θ , you should find that $\theta = 140.184$.

Solution 6.9

Let T denote the random time required to repair the machine. It is assumed to be exponentially distributed with mean $\theta_T = 2$. Let X denote the cost of replacement parts. It is assumed to be gamma distributed with parameters α and θ_X such that:

$$100 = E[X] = \alpha \theta_X$$

$$5,000 = \text{var}(X) = \alpha \theta_X^2$$

Solving these equations results in:

$$\alpha = 2 \quad \theta_X = 50$$

We are asked to calculate the probability of the event $\{T > 3\} \cup \{X > 150\}$.

We can calculate the two components as follows:

$$\Pr(T > 3) = 1 - \Pr(T < 3) = e^{-3/2} = 0.2231$$

$$\Pr(X > 150) = 1 - \Pr(X < 150) = e^{-150/50} \left(1 + \frac{150}{50} \right) = 0.1991$$

By the additive probability law, we have:

$$\begin{aligned} \Pr(\{T > 3\} \cup \{X > 150\}) &= \Pr(T > 3) + \Pr(X > 150) - \Pr(\{T > 3\} \cap \{X > 150\}) \\ &= \Pr(T > 3) + \Pr(X > 150) - \Pr(T > 3) \Pr(X > 150) \\ &= 0.2231 + 0.1991 - 0.2231 \times 0.1991 = 0.3778 \end{aligned}$$

Solution 6.10

We can calculate the required probability as follows:

$$\Pr(4 < S < 8) = \Pr(4 < S < 8 \cap N = 0) + \Pr(4 < S < 8 \cap N = 1) + \Pr(4 < S < 8 \cap N > 1)$$

If there are no claims (*ie* $N = 0$), then the claim amount must be zero, so:

$$\Pr(4 < S < 8 \cap N = 0) = 0$$

Therefore:

$$\Pr(4 < S < 8) = \Pr(4 < S < 8 \cap N = 1) + \Pr(4 < S < 8 \cap N > 1)$$

So from basic laws of probability, we have:

$$\begin{aligned} \Pr(4 < S < 8) &= \Pr(4 < S < 8 \cap N = 1) + \Pr(4 < S < 8 \cap N > 1) \\ &= \Pr(4 < S < 8 | N = 1) \times \Pr(N = 1) + \Pr(4 < S < 8 | N > 1) \times \Pr(N > 1) \\ &= \underbrace{(F(8) - F(4))}_{\text{exp, } \theta=5} \times \frac{1}{3} + \underbrace{(F(8) - F(4))}_{\text{exp, } \theta=8} \times \frac{1}{6} \\ &= \left((1 - e^{-8/5}) - (1 - e^{-4/5}) \right) \times \frac{1}{3} + \left((1 - e^{-8/8}) - (1 - e^{-4/8}) \right) \times \frac{1}{6} \\ &= 0.1223 \end{aligned}$$

Solution 6.11

The waiting time between accidents follows an exponential distribution with mean $\theta = \frac{5}{2} = 2.5$ days.

Hence:

$$\Pr(X > 3) = 1 - F(3) = e^{-\frac{3}{2.5}} = e^{-1.2} = 0.3012$$

Solution 6.12

From the form of the pdf, we can see that X follows a gamma distribution with $\alpha = 6$ and $\theta = 100$.

Hence:

$$E[X] = \alpha\theta = 6 \times 100 = 600$$

Solution 6.13

First, we'll determine the parameter values:

$$15 = E[X] = \alpha\theta$$

$$75 = \text{var}(X) = \alpha\theta^2$$

$$\Rightarrow \theta = \frac{\text{var}(X)}{E[X]} = 5 \Rightarrow \alpha = 3$$

Since α is a positive integer, the cdf can be written as:

$$F(x) = 1 - e^{-x/5} \left(1 + \frac{x}{5} + \frac{x^2}{2!5^2} \right)$$

Finally:

$$\Pr(X > 30 | X > 15) = \frac{\Pr(X > 30)}{\Pr(X > 15)} = \frac{1 - F(30)}{1 - F(15)} = \frac{1 - 0.93803}{1 - 0.57681} = \frac{0.06197}{0.42319} = 0.1464$$

Solution 6.14

From the form of the moment generating function, we can see that X follows a gamma distribution with parameters $\alpha = 2$ and $\theta = 3$. Hence:

$$\text{var}(X) = \alpha\theta^2 = 2 \times 3^2 = 18$$

Solution 6.15

Differentiating the cumulant generating function, we have:

$$R_X(t) = \ln M_X(t) = \ln\left((1-\theta t)^{-\alpha}\right) = -\alpha \ln(1-\theta t)$$

$$R'_X(t) = \frac{\alpha\theta}{(1-\theta t)}$$

$$R''_X(t) = \frac{\alpha\theta^2}{(1-\theta t)^2} \Rightarrow R''_X(0) = \alpha\theta^2$$

$$R'''_X(t) = \frac{2\alpha\theta^3}{(1-\theta t)^3} \Rightarrow R'''_X(0) = 2\alpha\theta^3$$

Hence the skewness is:

$$\frac{E\left[(X-\mu)^3\right]}{\sigma^3} = \frac{R'''_X(0)}{(R''_X(0))^{3/2}} = \frac{2\alpha\theta^3}{\alpha^{3/2}\theta^3} = \frac{2}{\sqrt{\alpha}}$$

Solution 6.16

⇒ From 2010, this question is no longer covered by the Exam P syllabus.

Solution 6.17

⇒ From 2010, this question is no longer covered by the Exam P syllabus.

Solution 6.18

First we need the value of c . This can be found from:

$$1 = \int_0^{\infty} c x^2 e^{-4x^3} dx$$

$$\Rightarrow \frac{1}{c} = \int_0^{\infty} x^2 e^{-4x^3} dx$$

Now:

$$\int g'(x) e^{g(x)} dx = e^{g(x)} + k$$

where k is a constant.

So:

$$\frac{1}{c} = -\frac{1}{12} \int_0^{\infty} -12x^2 e^{-4x^3} dx = -\frac{e^{-4x^3}}{12} \Bigg|_0^{\infty} = \frac{1}{12}$$

$$\Rightarrow c = 12$$

Now we need:

$$\Pr(X > 0.5) = \int_{0.5}^{\infty} 12x^2 e^{-4x^3} dx = -e^{-4x^3} \Bigg|_{0.5}^{\infty} = e^{-4(0.5)^3} = e^{-0.5} = 0.6065$$

Solution 6.19

⇒ From 2010, this question is no longer covered by the Exam P syllabus.

Solution 6.20

⇒ From 2010, this question is no longer covered by the Exam P syllabus.