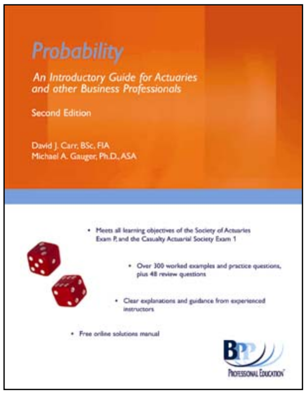

PROFESSIONAL EDUCATION

Probability, Second Edition

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Solutions to practice questions – Chapter 8

Solution 8.1

The required probability is:

$$\begin{aligned}\Pr(X+Y\le 3) &= f_{X,Y}(0,1)+f_{X,Y}(0,2)+f_{X,Y}(0,3)+f_{X,Y}(1,1)+f_{X,Y}(1,2)+f_{X,Y}(2,1)\\ &= 0.05+0.09+0.11+0.18+0.14+0.13\\ &= 0.7\end{aligned}$$

Solution 8.2

The marginal probability function of X is:

$$f_X(0)=\sum_{\text{all } y} f_{X,Y}(0,y)=f_{X,Y}(0,1)+f_{X,Y}(0,2)+f_{X,Y}(0,3)=0.05+0.09+0.11=0.25$$

$$f_X(1)=\sum_{\text{all } y} f_{X,Y}(1,y)=f_{X,Y}(1,1)+f_{X,Y}(1,2)+f_{X,Y}(1,3)=0.18+0.14+0.03=0.35$$

$$f_X(2)=\sum_{\text{all } y} f_{X,Y}(2,y)=f_{X,Y}(2,1)+f_{X,Y}(2,2)+f_{X,Y}(2,3)=0.13+0.08+0.04=0.25$$

$$f_X(3)=\sum_{\text{all } y} f_{X,Y}(3,y)=f_{X,Y}(3,1)+f_{X,Y}(3,2)+f_{X,Y}(3,3)=0.07+0.06+0.02=0.15$$

Solution 8.3

The marginal probability function of Y given that $X=2$ is:

$$f_Y(1|X=2)=\frac{f_{X,Y}(2,1)}{f_X(2)}=\frac{0.13}{0.25}=0.52$$

$$f_Y(2|X=2)=\frac{f_{X,Y}(2,2)}{f_X(2)}=\frac{0.08}{0.25}=0.32$$

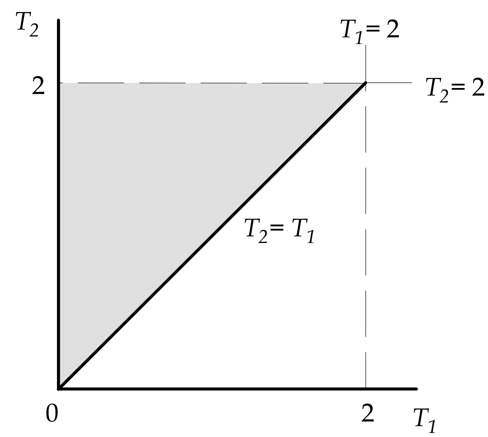
$$f_Y(3|X=2)=\frac{f_{X,Y}(2,3)}{f_X(2)}=\frac{0.04}{0.25}=0.16$$

Solution 8.4

We are given that the probability density function is:

$$f_{T_1,T_2}(t_1, t_2) = c \qquad \text{for } 0 \le t_1 \le t_2 \le 2$$

The region $0 \le T_1 \le T_2 \le 2$ in the $T_1 \times T_2$ plane is a triangle with area 2 (the shaded area in the figure on the right).



Since the double integral of the joint pdf over the $T_1 \times T_2$ plane must equal 1, then it follows that:

$$c = 0.5$$

We can then calculate $\Pr(2T_1 < T_2)$ as follows:

$$\Pr(2T_1 < T_2) = \int_{t_1=0}^{1} \int_{t_2=2t_1}^{2} 0.5 dt_2\, dt_1 = \int_{t_1=0}^{1} \left(0.5t_2\Big|_{2t_1}^{2}\right) dt_1$$

$$= \int_{t_1=0}^{1} (1-t_1)\, dt_1 = \left(t_1 - 0.5t_1^2\right)\Big|_{t_1=0}^{1}$$

$$= 0.5$$

If you cannot see why $c = 0.5$, then you will need to carry out one of the following integrations:

$$\int_{t_1=0}^{2} \int_{t_2=t_1}^{2} c\ dt_2\ dt_1 = 1 \ \textit{ or } \int_{t_2=0}^{2} \int_{t_1=0}^{t_2} c\ dt_1\ dt_2 = 1$$

To calculate the probability you could have carried out the following integral:

$$\int_{t_2=0}^{2} \int_{t_1=0}^{\frac{1}{2}t_2} 0.5\ dt_1\ dt_2$$

Solution 8.5

Let X be the time until next claim for a Basic Policy, and let Y be the time until next claim for a Deluxe Policy.

Since the two random variables are independent, the joint density is equal to the product of the two marginal densities. Hence:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{e^{-x/2}}{2} \times \frac{e^{-y/3}}{3} \qquad \text{for } x, y > 0$$

Thus, the probability that the next claim is from a Deluxe Policy is:

$$\Pr(Y < X) = \int_{x=0}^{\infty} \int_{y=0}^{x} f_{X,Y}(x,y)\, dy\, dx = \int_{x=0}^{\infty} \int_{y=0}^{x} \frac{e^{-x/2} e^{-y/3}}{2 \times 3} dy\, dx$$

$$= \int_{x=0}^{\infty} \frac{e^{-x/2}}{2} \left(-e^{-y/3} \Big|_{y=0}^{x} \right) dx = \int_{x=0}^{\infty} \frac{e^{-x/2}}{2} \left(1 - e^{-x/3}\right) dx$$

$$= \frac{1}{2} \int_{x=0}^{\infty} e^{-x/2} - e^{-5x/6}\ dx = \frac{1}{2} \left(-2e^{-x/2} + \frac{6}{5} e^{-5x/6} \Big|_{x=0}^{\infty} \right)$$

$$= \frac{1}{2} \left(-0 + 0 + 2 - \frac{6}{5} \right) = \frac{2}{5}$$

Solution 8.6

Let Y be the time to process a claim of amount X.

To calculate $\Pr(Y > 3)$, we must first identify the joint density function:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y|X=x)$$

$$= \frac{3x^2}{8} \times \frac{1}{x} = \frac{3x}{8} \qquad 0 \le x \le 2 \text{ and } x \le y \le 2x$$

Since Y is distributed uniformly on the interval $[x, 2x]$, then $\Pr(Y > 3) = 0$ if $X < 1.5$.

So, we can calculate $\Pr(Y > 3)$ as:

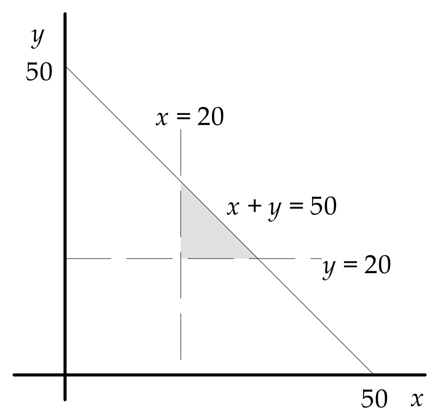
$$\Pr(Y > 3) = \iint f(x,y)\, dy\, dx = \int_{x=1.5}^{2} \int_{y=3}^{2x} \frac{3x}{8}\, dy\, dx$$

$$= \int_{1.5}^{2} \frac{3x}{8}(2x-3)\, dx = \frac{3}{8} \int_{1.5}^{2} 2x^2 - 3x\, dx$$

$$= \frac{3}{8} \times \left(\frac{2x^3}{3} - \frac{3x^2}{2} \right)\Bigg|_{1.5}^{2} = \frac{3}{8}\left(-0.6667 - (-1.125)\right)$$

$$= 0.1719$$

Solution 8.7



The best way to identify the critical region is to draw a graph.

The joint density function is positive over the region below the line $x + y = 50$, for $0 < x < 50$ and $0 < y < 50$. This is the large triangle in the figure on the right.

To calculate the probability that both components are still functioning at time 20, we need to integrate the joint density function over a smaller region, with $x > 20$ and $y > 20$. This is the smaller shaded triangle in the figure on the right.

Over what range of x do we need to integrate the joint density function?

From the graph, it should be clear that we need to integrate the joint density function over the range $20 < x < 30$.

And over what range of y do we need to integrate the joint density function?

The two constraints on y are $y > 20$ and $x + y < 50$, so it follows that we integrate over the range $20 < y < 50 - x$.

Hence the required probability is:

$$\frac{6}{125{,}000} \int_{20}^{30} \int_{20}^{50-x} (50 - x - y)\, dy\, dx$$

Solution 8.8

The conditional probability function of X given that $Y=1$ is:

$$f_X\left(0\middle|Y=1\right)=\frac{f_{X,Y}(0,1)}{f_Y(1)}=\frac{0.1}{0.6} \qquad f_X\left(1\middle|Y=1\right)=\frac{0.2}{0.6} \qquad f_X\left(2\middle|Y=1\right)=\frac{0.3}{0.6}$$

Hence:

$$E\left[X\middle|Y=1\right]=0\times\frac{0.1}{0.6}+1\times\frac{0.2}{0.6}+2\times\frac{0.3}{0.6}=\frac{4}{3}=1.333$$

Solution 8.9

We want to calculate $E\left[Y|X=50\right]$. The first step is to calculate the conditional probability density function:

$$f_X(50)=\int_0^{50} f(50,y)\,dy=\int_0^{50}\frac{8\times 50\times y}{100^4}\,dy=\frac{8\times 50^3}{2\times 100^4}$$

$$f_Y\left(y|X=50\right)=\frac{f(50,y)}{f_X(50)}=\frac{8\times 50\times y}{100^4}\bigg/\frac{8\times 50^3}{2\times 100^4}=\frac{2y}{50^2} \qquad \text{for } 0\le y\le 50$$

Hence:

$$E\left[Y|X=50\right]=\int_0^{50} y\,f_Y\left(y|X=50\right)dy=\int_0^{50} y\times\frac{2y}{50^2}\,dy$$

$$=\left(\frac{2\times y^3}{3\times 50^2}\right)\Bigg|_0^{50}=\frac{2\times 50^3}{3\times 50^2}=33.33$$

Solution 8.10

From the given joint probabilities we can compute the following:

$$\Pr\left(Y=0\middle|X=1\right)=\frac{\Pr\left(X=1,Y=0\right)}{\Pr\left(X=1\right)}=\frac{0.050}{0.050+0.125}=\frac{2}{7}$$

$$\Pr\left(Y=1\middle|X=1\right)=1-\frac{2}{7}=\frac{5}{7}$$

We can now compute the conditional variance as follows:

$$E\left[Y\middle|X=1\right]=0\times\frac{2}{7}+1\times\frac{5}{7}=\frac{5}{7}$$

$$E\left[Y^2\middle|X=1\right]=0^2\times\frac{2}{7}+1^2\times\frac{5}{7}=\frac{5}{7}$$

$$\text{var}\left(Y\middle|X=1\right)=\frac{5}{7}-\left(\frac{5}{7}\right)^2=\frac{10}{49}=0.2041$$

Solution 8.11

We want to calculate $\text{var}(Y|Y>25)$. The first step is to calculate the conditional probability density function.

The probability density function of Y is:

$$f_Y(y) = 0.01 \text{ for } 0 \le y \le 100$$

and the probability that $Y > 25$ is:

$$\Pr(Y>25) = \int_{25}^{100} 0.01\, dy = 0.01y\Big|_{25}^{100} = 0.75$$

So the conditional density is given by:

$$f_Y(y|Y>25) = \frac{f_Y(y)}{\Pr(Y>25)} = \frac{0.01}{0.75} = \frac{1}{75} \qquad \text{for } 25 < y \le 100$$

We can recognize the pdf of $y|Y>25$ as uniform on $[25,100]$, hence the variance is:

$$\text{var}(Y|\, Y>25) = \frac{(100-25)^2}{12} = 468.75$$

Solution 8.12

Using property (3) of the joint moment generating function:

$$M_{X+Y}(t) = M_{X,Y}(t,t)$$

Hence, the moment generating function of $X+Y$ is:

$$M_{X+Y}(t) = M_{X,Y}(t,t) = \int_{y=0}^{1} \int_{x=0}^{y} e^{tx+ty} f_{X,Y}(x,y)\, dx\, dy$$

$$= 2\int_{y=0}^{1} e^{ty} \int_{x=0}^{y} e^{tx}\, dx\, dy$$

$$= \frac{2}{t} \int_{y=0}^{1} e^{ty}\left(e^{ty}-1\right) dy$$

$$= \frac{2}{t}\left(\frac{e^{2ty}}{2t} - \frac{e^{ty}}{t}\right)\Bigg|_0^1$$

$$= \frac{e^{2t} - 2e^{t} + 1}{t^2}$$

Solution 8.13

We can use the double expectation theorem:

$$E[X] = E\left[E\left[X|Y\right]\right]$$

Since the mean of a gamma distribution is $\alpha\theta$, the expected value of X given Y is:

$$E\left[X|Y\right] = Y \times 10$$

Since Y follows a uniform distribution on the interval $(5,8)$, we have:

$$E[Y] = 6.5$$

Hence, by the double expectation theorem, we have:

$$E[X] = E\left[E\left[X|Y\right]\right] = E[Y \times 10] = 10E[Y] = 10 \times 6.5 = 65$$

Solution 8.14

Let Y be the total claim amount based on $N = n$ independent claims, *ie*:

$$Y = X_1 + X_2 + \cdots + X_n$$

Then we have:

$$E\left(Y|N\right) = N \times E[X_i] = 250N$$
$$\text{var}\left(Y|N\right) = N \times \text{var}(X_i) = N \times 40^2 = 1{,}600N$$

Using Theorem 8.7, we have:

$$\begin{aligned}\text{var}(Y) &= E\left[\text{var}\left(Y|N\right)\right] + \text{var}\left(E\left[Y|N\right]\right) \\ &= E[1{,}600N] + \text{var}(250N) \\ &= 1{,}600E[N] + 250^2\,\text{var}(N) \\ &= 1{,}600 \times 200 + 250^2 \times 200 = 12{,}820{,}000\end{aligned}$$

Hence the standard deviation is $\sqrt{12{,}820{,}000} = 3{,}581$.

Solution 8.15

We have:

$$\begin{aligned}\text{var}(4X - Y + 80) &= \text{var}(4X - Y) \\ &= 4^2\,\text{var}(X) + (-1)^2\,\text{var}(Y) + (2)(4)(-1)\,\text{cov}(X,Y) \\ &= (16)(8^2) + (1)(10^2) + (-8)(-20) \\ &= 1{,}284\end{aligned}$$

So the standard deviation is $\sqrt{1{,}284} = 35.83$.

Solution 8.16

We'll calculate the covariance using the following result:

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y]$$

We have:

$$E[X] = 0\times 0.25 + 1\times 0.35 + 2\times 0.25 + 3\times 0.15 = 1.3$$
$$E[Y] = 1\times 0.43 + 2\times 0.37 + 3\times 0.20 = 1.77$$

and:

$$E[XY] = 0\times 0.25 + 1\times 0.18 + 2\times 0.27 + 3\times 0.10 + 4\times 0.08 + 6\times 0.10 + 9\times 0.02 = 2.12$$

Finally:

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y] = 2.12 - 1.3\times 1.77 = -0.181$$

Solution 8.17

From the definition of the correlation coefficient:

$$\rho = \frac{\text{cov}(X,Y)}{\sigma_X\,\sigma_Y}$$
$$\Rightarrow\ \sigma_X\,\sigma_Y = \frac{\text{cov}(X,Y)}{\rho} = \frac{24}{0.8} = 30$$

We are also told that:

$$\sigma_X^2 = 4\sigma_Y^2$$
$$\Rightarrow\ \sigma_Y = 0.5\sigma_X$$
$$\Rightarrow\ \sigma_X^2 = \frac{30}{0.5} = 60$$

Solution 8.18

The covariance is calculated as:

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y]$$

From the question, we know that:

$$f_X(x) = \frac{1}{12} \quad \text{for } 0 \le x \le 12$$

$$f_Y(y|X=x) = \frac{1}{x} \quad \text{for } 0 \le y \le x$$

$$f(x,y) = f_X(x)\,f_Y(y|X=x) = \frac{1}{12x} \quad \text{for } 0 \le y \le x \le 12$$

We can calculate $E[XY]$ as:

$$E[XY]=\iint xy\,f(x,y)\,dy\,dx=\int_{x=0}^{12}\int_{y=0}^{x}xy\frac{1}{12x}\,dy\,dx$$

$$=\int_{x=0}^{12}\int_{y=0}^{x}\frac{y}{12}\,dy\,dx=\frac{1}{24}\int_{x=0}^{12}\left(y^2\Big|_{y=0}^{x}\right)dx$$

$$=\frac{1}{24}\int_{0}^{12}x^2\,dx=\frac{x^3}{72}\Bigg|_0^{12}=24$$

Since X is uniformly distributed on $[0,12]$, we have:

$$E[X]=\frac{12}{2}=6$$

And since $Y|X=x$ is uniformly distributed on $[0,x]$, we have:

$$E[Y]=E\Big[E\big[Y|X\big]\Big]=E\left[\frac{X}{2}\right]=\frac{1}{2}\times E[X]=3$$

Finally:

$$\text{cov}(X,Y)=E[XY]-E[X]E[Y]=24-6\times 3=6$$

Solution 8.19

We have:

$$E\big[X|Y=17\big]=\mu_X+\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)=12+0.8\times\frac{3}{5}\times(17-20)=10.56$$

Solution 8.20

The conditional distribution of Y given that $X=x$ is:

$$Y|X=x\sim N\left(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\ ,\ \sigma_Y^2(1-\rho^2)\right)$$

Hence, the conditional distribution of Y given that $X=15$ is:

$$Y|X=15\sim N\left(20+0.8\times\frac{5}{3}\times(15-12)\ ,\ 25(1-0.8^2)\right)=N\left(24\ ,3^2\right)$$

So, the required probability is:

$$\Pr\left(Y<30|X=20\right)=\Phi\left(\frac{30-24}{3}\right)=\Phi(2)=0.9772$$

