Lesson 1

Limits and Continuity of Functions

Overview

In this first calculus lesson, we will study how the value of a function $f(x)$ changes as x approaches a particular number a. We'll consider whether or not the value of the function approaches a limiting value, and if it does, we'll learn how to calculate this limit.

We'll also use this theory to identify whether or not a function is continuous.

Although it's unusual for Course 1 exam questions to test this material solely, the topics covered here form the foundation of many of the other calculus lessons, *eg* differentiation and power series. We strongly recommend that you spend adequate time on this lesson to gain a deep understanding of the fundamental ideas so that you can make good progress through the remainder of these lessons.

BPP Learning Objectives

This lesson covers the following BPP learning objectives:

(C1) *Identify whether a limit of a function exists or fails to exist.*

(C2) *Calculate the limit of a function (where it exists) using first principles and limit rules, including right-hand and left-hand limits.*

(C3) *Identify whether a function is continuous at a given point.*

Theory

Functions

Definitions

y is a **function** of x if for each x in a given set of numbers D (the **domain**) there is a unique number y assigned to x.

The standard notation is:

$y = f(x)$ (This is read as "y is equal to f of x")

The **range** of the function $y = f(x)$ is the set of all values of $f(x)$ as x varies over the domain.

The **graph** of $y = f(x)$ is a plot of all pairs $(x, f(x))$ as x varies over the domain. It is a pictorial representation of the relation between x and y.

For example, consider the linear function $y = 1 + 2x$. The domain of this function is $\mathbb{R}$ (the set of real numbers). The range is also $\mathbb{R}$, since for any given number y we can find a value of $x \in \mathbb{R}$ such that $y = 1 + 2x$. The graph of this function is a straight line.

Note: If the domain of a function is not stated explicitly, it is implicit that the domain is considered to be the largest set of numbers for which the definition of the function makes sense. For example, for the function $y = 1/\sqrt{1-x}$, then the domain consists of all numbers $x < 1$, since we cannot divide by zero or calculate a square root of a negative number.

Special types of functions

A **polynomial** function (of degree n) is one that can be written in the form:

$$y = p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where the coefficients $a_0, a_1, \cdots, a_n$ are real numbers, the degree n is a positive integer and $a_n \neq 0$.

A **rational** function is a ratio of polynomial functions, *ie*:

$$y = \frac{p_1(x)}{p_2(x)}$$ where $p_1(x)$ and $p_2(x)$ are polynomial functions

An **algebraic** function can be written as a formula employing constants, the variable x, and the operations of addition, subtraction, multiplication, division, and extraction of an n^{th} root (n a positive integer), *eg*:

$$y = \sqrt{\frac{x^2 - 2x}{1 + x}}$$

Algebraic functions include rational functions, which in turn include polynomial functions.

Theory

Spliced functions are frequently used in economics and insurance applications, and often appear on the Course 1 exam to test conceptual understanding of limits, continuity, and differentiability. The idea behind this type of function is that different formulas apply to different segments of the domain.

For example, suppose that x is an individual's taxable annual income, and that $y = f(x)$ is the total state tax payable by that individual. Suppose that no tax is payable on the first 10,000 of taxable income, a 5% tax is payable on the next 40,000 of taxable income, and a tax of 6.25% is payable on any taxable income in excess of 50,000. Then:

$$y = f(x) = \begin{cases} 0 & \text{if } 0 \le x < 10{,}000 \\ 0.05(x - 10{,}000) & \text{if } 10{,}000 \le x < 50{,}000 \\ 2{,}000 + 0.0625(x - 50{,}000) & \text{if } 50{,}000 \le x \end{cases}$$

Note that the constant of 2,000 in the third line is equal to the tax paid on the first 50,000 (*ie* 5% of 40,000).

The limit of a function

Suppose $y = f(x)$ is defined near the point $x = a$, but perhaps not at $x = a$ itself (*ie* the domain includes the set $(a - \delta, a) \cup (a, a + \delta)$ for some $\delta > 0$).

The limit of $f(x)$ as x approaches a is denoted $\lim_{x \to a} f(x)$.

Intuitively, $f(x)$ *gets closer to* this limit as *x gets closer to a, but not equal to a.*

For example, as x gets near to 1 (but not equal to 1), the expression $2x^2 + 3$ gets closer to 5 since $2x^2$ will be near 2. We would write this as:

$$\lim_{x \to 1} (2x^2 + 3) = 5$$

The precise definition is as follows:

Definition

Let $\lim_{x \to a} f(x)$ denote the **limit of a function** $f(x)$ as x approaches a.

Then $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ (depending on ε) such that:

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

In words, if $\lim_{x \to a} f(x) = L$ then $f(x)$ is near L for x near a (but not equal to a).

Importantly, a limit may fail to exist. This definition is not used as a device to calculate limits, but it is the starting point for the proof of the following theorem.

Theory

Theorem

Basic Limits: (i) $\lim_{x \to a} c = c$

(ii) $\lim_{x \to a} x = a$

Limit Rules: If $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$, then:

(i) $\lim_{x \to a} (f(x) \pm g(x)) = L_1 \pm L_2$

(ii) $\lim_{x \to a} \left(f(x)g(x)\right) = L_1 \times L_2$

(iii) $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{L_1}{L_2}$ if $L_2 \neq 0$ This is known as the **quotient rule**.

(iv) $\lim_{x \to a} \left(f(x)\right)^{1/n} = L_1^{1/n}$ for a positive integer n

Once these basic limits and limit rules are proven, a large class of limit problems becomes routine. For example, we have the following results for polynomial functions:

- $\lim_{x \to a} p(x) = p(a)$ if $p(x)$ is a polynomial function
- $\lim_{x \to a} \left(\frac{p_1(x)}{p_2(x)}\right) = \frac{p_1(a)}{p_2(a)}$ if $p_1(x)$ and $p_2(x)$ are polynomial functions and $p_2(a) \neq 0$

Notice that these rules say that the limit is simply the function value. We will see shortly that this is the idea of continuity of the function $f(x)$ at the point $x = a$.

Let's take a look – using graphs to illustrate the ideas – at the typical ways in which limits may exist or fail to exist.

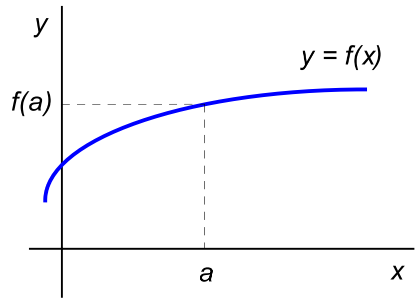
Theory

Limits that exist as x approaches a

In the figure below, the graph is unbroken at $x = a$. The limit exists and is equal to the function value, *ie*:

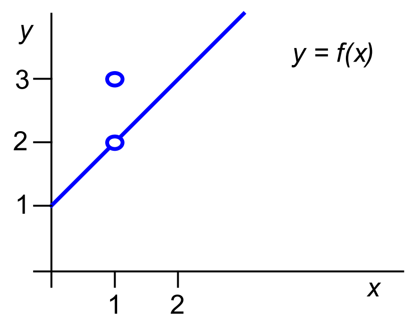
$$\lim_{x \to a} f(x) = f(a)$$

This is typical of a polynomial function.



The next figure is a graph of the function:

$$y = f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & x \neq 1 \\ 3 & x = 1 \end{cases}$$



Note that for $x \neq 1$, $f(x) = x + 1$.

The graph of this function is the line $y = x + 1$ with a “hole” at the point $(1,2)$ and an “extra point” above the line at $(1,3)$ instead. The function still has a limit of 2 as x approaches 1 (but is not equal to 1).

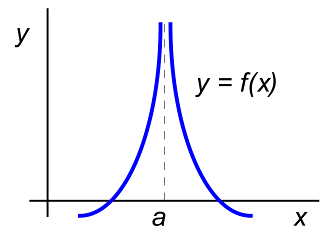
So, although the limit exists, in this case:

$$\lim_{x \to 1} f(x) \neq f(1)$$

It is important to understand that in this example the value of the function at $x = a$ is irrelevant as far as the definition of $\lim_{x \to a} f(x)$ is concerned.

Theory

Limits that fail to exist as x approaches a



In the figure to the right, the limit fails to exist because the function increases without bound as x approaches a.

The graph has the line $x = a$ as a **vertical asymptote**.

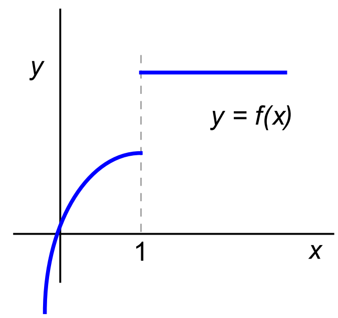
For functions such as this, we sometimes use the notation:

$$\lim_{x \to a} f(x) = \infty$$

but strictly speaking the limit does not exist. A limit must be a real number, not ∞.

This picture is typical of functions such as $y = \dfrac{1}{(x-a)^2}$.

In the next figure on the right, the limit fails to exist due to a **jump discontinuity** (*ie* a break in the graph).



A spliced function such as the following one would have this type of graph:

$$y = f(x) = \begin{cases} 2x - x^2 & \text{if } x \le 1 \\ 2 & \text{if } x > 1 \end{cases}$$

As x approaches 1 from the left side, $f(x)$ approaches a value of $2(1) - 1^2 = 1$.

On the other hand, as x approaches 1 from the right side, $f(x)$ is always 2.

So, there is no number L that the function is near for all x near 1 (but not equal to 1).

Another characteristic of a graph that results in a limit failing to exist is the idea of **very rapid oscillation** as x approaches a. The classic example of this behavior is the limit:

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

The portion of the graph of this function for $0 < x \le 1$ can be obtained from the sine wave $y = \sin(x)$, $x \ge 1$ by reversing and compressing it. Imagine the infinitely many waves being compressed to a finite interval. The closer you get to zero, the more rapidly the graph of $y = \sin(1/x)$ oscillates between $+1$ and -1. The limit of $y = \sin(1/x)$ as x approaches 0 cannot exist due to this rapid oscillation. If you draw this graph with a graphing calculator, you will see this graph appear as a solid rectangle near the Y axis.

Theory

Right-hand and left-hand limits

Definition

The symbol $\lim_{x \to a^+} f(x)$ is called a **right-hand limit**.

This expression refers to the limit of $f(x)$ for x near to a and **greater** than a.

The symbol $\lim_{x \to a^-} f(x)$ is called a **left-hand limit**.

This expression refers to the limit of $f(x)$ for x near to a and **less** than a.

In general, we have the following theorem:

Theorem

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

For example, consider the spliced function given by:

$$y = f(x) = \begin{cases} 2x - x^2 & \text{if } x < 2 \\ x + c & \text{if } x \geq 2 \end{cases}$$

Is there a way to choose c so that $\lim_{x \to 2} f(x)$ exists?

Since $2x - x^2$ is a polynomial, it is easy to see that:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2x - x^2 = \lim_{x \to 2} 2x - x^2 = 2(2) - 2^2 = 0$$

Similarly, we have:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x + c = 2 + c$$

The left and right hand limits are equal if $c = -2$.

Theory

Infinite limits

The expression $\lim_{x \to a} f(x) = \infty$ means that the limit fails to exist because the function increases without bound as x approaches a.

For example, we have:

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Similarly, the expression $\lim_{x \to a^+} f(x) = -\infty$ means that the right hand limit fails to exist because the function decreases without bound as x approaches a from the right.

For example, we have:

$$\lim_{x \to 2^+} \frac{1}{2-x} = -\infty .$$

You'll need to be comfortable with these applying ideas to other situations in the Course 1 exam. We'll meet many more limit problems during the other lessons.

Limits at infinity

The expression $\lim_{x \to \infty} f(x) = L$ means that $f(x)$ approaches L as x increases without bound.

For example, we have:

$$\begin{aligned}\lim_{x \to \infty} \frac{x}{\sqrt{2x^2 + 1000}} &= \lim_{x \to \infty} \frac{x}{x\sqrt{2 + 1000/x^2}} \\ &= \lim_{x \to \infty} \frac{1}{\sqrt{2 + 1000/x^2}} \\ &= \frac{1}{\sqrt{2+0}} \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

Similarly, the expression $\lim_{x \to -\infty} f(x) = L$ means that $f(x)$ approaches L as x decreases without bound.

Theory

Continuity of a function

The basic idea of a continuous function is that its graph is unbroken, except where breaks in the domain occur. A continuous function can be drawn without lifting your pencil from the paper, except when you come to a break in the domain. Here is the formal definition:

Definition

The function $y = f(x)$ is **continuous** at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

The function $y = f(x)$ is a **continuous function** if it is continuous at every point in its domain.

The function $y = f(x)$ is **right-continuous** at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

The function $y = f(x)$ is **left-continuous** at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

Three criteria must be met for a function to be continuous at a single point $x = a$:

1. $f(x)$ is defined at $x = a$, *ie* a is in the domain of $y = f(x)$.
2. The function limit as x approaches a must exist.
3. The limit is equal to the function value.

We now have the following result:

Theorem

An algebraic function is continuous at every point in its domain.

For example, the function $y = f(x) = \dfrac{x^2+1}{x-1}$ is defined for all $x \neq 1$.

Hence the domain is $D = (-\infty, 1) \cup (1, \infty)$, a union of two intervals.

It is a rational function, so it is continuous on its domain. Its graph must thus consist of two unbroken curves. The break between these two pieces occurs at $x = 1$, the break in the domain. So, we have:

$$\lim_{x \to a} f(x) = f(a) \quad \text{for all } a \neq 1$$

And the limits at the break in the domain are:

$$\lim_{x \to 1^+} f(x) = \infty \qquad \lim_{x \to 1^-} f(x) = -\infty$$

Worked examples

Example 1.1

Reference: BPP

Calculate the limit of $y = \dfrac{x^3 - 1}{x^2 - 1}$ as x approaches 1.

(A) 1.0
(B) 1.5
(C) 2.0
(D) 2.5
(E) The limit does not exist.

Solution

It's not initially clear whether or not the limit exists, since:

$$\lim_{x \to 1}\left(x^3 - 1\right) = 0 \quad \text{and} \quad \lim_{x \to 1}\left(x^2 - 1\right) = 0$$

So, we can't use the quotient rule in this case.

We can factorize the expression:

$$f(x) = \frac{x^3 - 1}{x^2 - 1} = \frac{(x-1)\left(x^2 + x + 1\right)}{(x-1)(x+1)} = \frac{x^2 + x + 1}{x+1} = h(x),\ x \neq 1$$

Since $f(x)$ and $h(x)$ are equal except at $x = 1$, it follows that:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} h(x).$$

Furthermore, $h(x)$ is a rational function that is defined at $x = 1$.

Hence, we have:

$$\lim_{x \to 1} h(x) = h(1) = \frac{3}{2} = 1.5$$

So, the correct answer is **B**.

Note 1: In the exam, you may like to check your work quickly using a calculator. For example, you could value the function at values of x just below 1 and just above 1 and try to identify a pattern. In this case, $f(1.001) = 1.50075$ and $f(0.999) = 1.49925$. While this is far from rigorous, it should give you confidence that the answer of 1.5 is correct.

Note 2: In Calculus Lesson 3, we'll introduce another method (called L'Hopital's Rule) that can often provide a much more efficient way to deal with the limit of a quotient when the limit of the denominator and numerator are both equal to zero.

Worked examples

Example 1.2

Reference: BPP

The following function is continuous at $x=1$:

$$y=f(x)=\begin{cases}\dfrac{x^3-1}{x^2-1} & \text{if } x<1\\ \sqrt{\dfrac{x+c}{2}} & \text{if } x\geq 1\end{cases}$$

Calculate the value of c.

(A) 1.5
(B) 2.0
(C) 2.5
(D) 3.0
(E) 3.5

Solution

From the theorem on page 7, it suffices to find a value of c so that:

$$\lim_{x\to 1^+} f(1)=\lim_{x\to 1^-} f(1)$$

Now:

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{x^3-1}{x^2-1}=1.5 \qquad \text{from Worked Example 1.1 on the previous page}$$

And:

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\sqrt{\frac{x+c}{2}}$$

Assuming that $c>-1$, and since an algebraic function is continuous at every point in its domain, we have:

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\sqrt{\frac{x+c}{2}}=\lim_{x\to 1}\sqrt{\frac{x+c}{2}}=\sqrt{\frac{1+c}{2}}$$

For the left and right-hand limits to be equal we must have:

$$1.5=\sqrt{\frac{1+c}{2}} \quad\Rightarrow\quad c=3.5$$

Since this does not contradict the assumption that $c>-1$ we are done.

So, the correct answer is **E**.

Practice questions

Question 1.1

Reference: BPP

Which of the following statements about limits is/are correct?

I. $\lim_{x\to 1} \frac{x^2+1}{2x-2} = \infty$

II. $\lim_{x\to 1^+} \frac{x^2-x}{x^2-2x+1} = 0$

III. $\lim_{x\to -\infty} \frac{\sqrt{|x|^{1.5}+10{,}000}}{2x} = 0$

(A) I only
(B) II only
(C) III only
(D) II and III only
(E) I and III only

Question 1.2

Reference: BPP

Suppose that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.

Which of the following statements is true?

(A) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ is either ∞ or $-\infty$

(B) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ does not exist

(C) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ is 0.

(D) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ may exist or it may not exist

(E) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = 2$ if $\frac{f(a+0.0001)}{g(a+0.0001)} = 1.9997$ and $\frac{f(a-0.0001)}{g(a-0.0001)} = 2.0002$

Practice questions

Question 1.3

Reference: BPP

Suppose that x is the taxable income of an individual and that the tax due on the taxable income is given by:

$$y = f(x) = \begin{cases} 0 & \text{if } x < 5000 \\ 0.05x - 250 & \text{if } 5000 \le x < 20{,}000 \\ rx - c & \text{if } 20{,}000 \le x \end{cases}$$

Determine the values of c and r such that the function is continuous and the tax on a taxable income of 50,000 is equal to 3,000.

(A) $c = 500$ and $r = 0.065$

(B) $c = 500$ and $r = 0.075$

(C) $c = 750$ and $r = 0.065$

(D) $c = 750$ and $r = 0.075$

(E) $c = 1{,}000$ and $r = 0.075$

Question 1.4

Reference: BPP

Determine the value of the constant c such that the following function is continuous at $x = 1$:

$$f(x) = \begin{cases} \sqrt{(c-x)^2 - 1} & \text{if } x < 1 \\ \dfrac{cx^2 + x - 2}{x^2 - x + 3} & \text{if } x \ge 1 \end{cases}$$

(A) $c = 1 + \sqrt{9/8}$

(B) $c = 1 - \sqrt{9/8}$

(C) $c = 1 \pm \sqrt{9/8}$

(D) $c = 1 + 1.5\sqrt{2}$

(E) No such c exists.