Lesson 2

The Derivative of a Function

Overview

In this lesson, we'll consider the derivative function, $f'(x)$, of a function $y = f(x)$.

The quantity $f'(x)$ represents the instantaneous rate of change of $y = f(x)$ with respect to changes in x. In other words, if x changes by a small amount, the derivative will tell us how the value of $f(x)$ will change.

The derivative also has a geometric interpretation. The derivative valued at $x = a$ is the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.

In this lesson we'll focus on the interpretation and calculation of the derivative. In particular, we'll study several important rules that will help us to calculate the derivative of (*ie* differentiate) common functions. In the next lesson, we'll study the practical applications of the derivative.

BPP Learning Objectives

This lesson covers the following BPP learning objectives:

(C4) *Understand the concept of differentiability and apply it to practical models.*

(C5) *Calculate the derivative of common functions using standard derivative rules.*

(C6) *Calculate derivatives using implicit differentiation.*

Theory

The derivative

Suppose that the function $y = f(x)$ is defined near the point $x = a$.

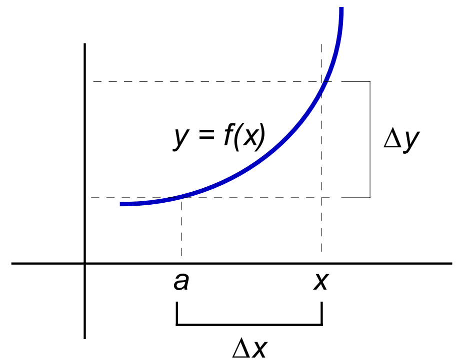
When x starts to move away from the current value a, how rapidly does $y = f(x)$ change?

For this purpose we introduce the following notation:

$$\Delta x = x - a$$
$$\Delta y = f(x) - f(a) = f(a + \Delta x) - f(a)$$

So, Δx is the change in x and Δy is the resulting change in y.

These quantities are shown in the figure on the right.



The ratio $\Delta y / \Delta x$ represents the **average rate of change of *y* with respect to *x*** as x varies over the interval $[a, a + \Delta x]$.

Note: The notation here indicates that $\Delta x > 0$, but Δx can be both positive and negative.

The idea of the **derivative** is to measure the **instantaneous rate of change of *y* with respect to *x*** by measuring the average rate of change over ever shorter intervals, *ie* we let Δx approach zero and take a limit.

Definition

The **derivative** of $y = f(x)$ with respect to x at $x = a$, denoted $f'(a)$, is defined as:

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

We write $f'(x)$ to denote the **derivative function** at a general x. The domain of this function is the set of all $x = a$ in the domain of $y = f(x)$ for which the limit of $\Delta y / \Delta x$ exists. You may also see the alternative notation:

- $\dfrac{dy}{dx}$ is often used as an alternative notation for $f'(x)$
- $\left.\dfrac{dy}{dx}\right|_{x=a}$ is often used as an alternative to $f'(a)$.

For example, let's calculate the derivative of the parabolic function $y = f(x) = x^2$:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{(2x)(\Delta x) + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x + 0 = 2x$$

Theory

Differentiability (smoothness) of a function

Suppose that we have a function $y = f(x)$ defined on an interval (a,b).

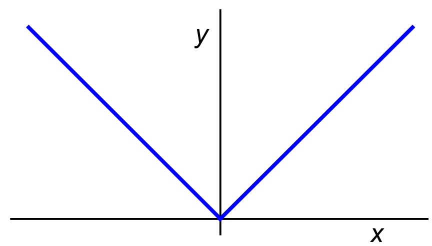
Continuity of this function on the interval can be intrepreted to mean that its graph is unbroken or connected.

The function is said to be **differentiable** on the interval if the derivative exists at each point in the interval.

If $y = f(x)$ is differentiable at $x = a$, then it is also continuous at $x = a$. However, the reverse statement is not generally true.

Let's explain why with an example, which will also convey the idea that a derivative fails to exist when there is a lack of "smoothness".

Consider the absolute value function $y = f(x) = |x|$. The graph of this function is shown in the figure on the right.



The graph of this function is connected, *ie* the function is continuous.

However, it is not "smooth" (*ie* it is not differentiable) at $x = 0$.

Computing the derivative of this function at $x = 0$ would require computing the following limit:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$$

But the ratio $\frac{|x|}{x}$ is equal to 1 if $x > 0$ and is equal to -1 if $x < 0$.

In other words:

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} -1 = -1$$

Since the two-sided limit fails to exist (because the one-sided limits are not equal), $f'(0)$ does not exist.

The function is not differentiable at $x = 0$ because of the "corner" (or lack of smoothness) at the origin.

However, it is differentiable everywhere else:

- $f'(x) = 1$ if $x > 0$
- $f'(x) = -1$ if $x < 0$.

Theory

Let's summarize the links between continuity (connectedness) and differentiability (smoothness) of a function

Continuity and differentiability – Summary

(1) $f(x)$ differentiable at $x=a$ $\Rightarrow$ $f(x)$ continuous at $x=a$

(2) $f(x)$ continuous on (a,b) $\Leftrightarrow$ the graph is unbroken

(3) $f(x)$ differentiable on (a,b) $\Leftrightarrow$ the graph is unbroken and smooth

Ways in which a derivative can fail to exist at a point

The derivative $f'(a)$ will fail to exist if:

- $f(x)$ is not continuous at $x=a$ (see Figure 1 below)
- $y=f(x)$ is continuous at $x=a$, but the graph has a "corner" at $x=a$ (see Figure 2 below)
- $y=f(x)$ is continuous at $x=a$, but the graph has a "cusp" at $x=a$ (see Figure 3 below)

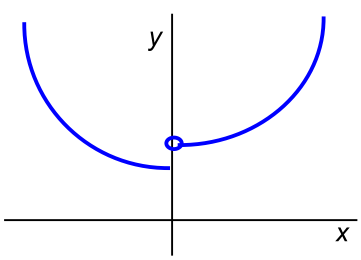


Figure 1
Discontinuity at $x = 0$

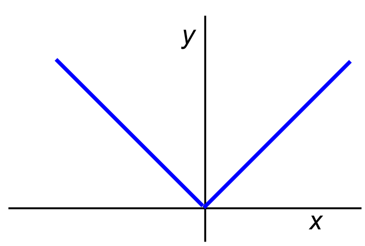


Figure 2
Corner at $x = 0$

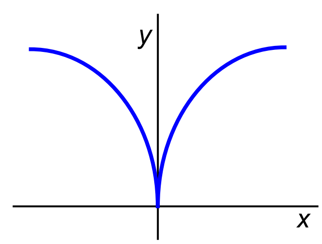


Figure 3
Cusp at $x = 0$

The tangent slope interpretation of the derivative

At the start of this chapter, we said that the derivative measures the instantaneous rate of change of y with respect to x. There is another important interpretation of the derivative.

The derivative $f'(a)$ can be interpreted as the **slope of the tangent line** to the function $f(x)$ at $x=a$.

Using the slope $f'(a)$ and the point $(a,f(a))$, together with the point-slope formula for a line, the equation of the tangent line to the function $f(x)$ at $x=a$ is:

$$y_{\text{tan}} = f(a) + f'(a)(x-a)$$

Theory

Derivative rules

In the Course 1 exam, you may be expected to calculate the derivative of several different types of functions, so you'll need to know the following basic rules. You should know most of these, if not all, already.

Derivative rules – Summary

(i) $c' = 0$

(ii) $\left(x^p\right)' = p\,x^{p-1}$ (any power $p \neq 0$)

(iii) $\left(c\,f(x)\right)' = c\,f'(x)$

(iii) $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

(iv) $\left(f(x)g(x)\right)' = f(x)g'(x) + f'(x)g(x)$ This is known as the **product rule**.

(v) $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$ This is known as the **quotient rule**.

(vi) $\left(g\left(f(x)\right)\right)' = g'\left(f(x)\right)f'(x)$ This is known as the **chain rule**.

(vii) $\left(\left(f(x)\right)^p\right)' = p\left(f(x)\right)^{p-1} f'(x)$ This is known as the **power rule**.

(viii) $\left(a^x\right)' = a^x \ln(a)$

For example, how can we calculate the derivative of:

$$y = \frac{x^3 + 2x^2}{x^2 + 1}$$

Using the quotient rule, let $f(x) = x^3 + 2x^2$ and $g(x) = x^2 + 1$, then:

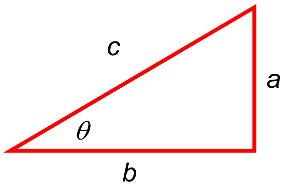
$$\frac{dy}{dx} = \frac{\left(x^2+1\right)\left(x^3+2x^2\right)' - \left(x^3+2x^2\right)\left(x^2+1\right)'}{\left(x^2+1\right)^2} = \frac{\left(x^2+1\right)\left(3x^2+4x\right) - \left(x^3+2x^2\right)(2x)}{\left(x^2+1\right)^2}$$

To complete the calculation, we would expand the products, combine terms where possible, and cancel common factors in the numerator and denominator if possible.

Theory

Derivatives of trignometric functions

We'll remind ourselves of the definitions of the basic trigonometric functions, and then summarize their derivatives:



$$\sin(\theta) = \frac{a}{c} \qquad \cos(\theta) = \frac{b}{c} \qquad \tan(\theta) = \frac{a}{b}$$

$$\csc(\theta) = \frac{c}{a} \qquad \sec(\theta) = \frac{c}{b} \qquad \cot(\theta) = \frac{b}{a}$$

Trigonometric derivatives – Summary

$$\left(\sin(\theta)\right)' = \cos(\theta) \qquad \left(\cos(\theta)\right)' = -\sin(\theta) \qquad \left(\tan(\theta)\right)' = \sec^2(\theta)$$

$$\left(\csc(\theta)\right)' = -\csc(\theta)\cot(\theta) \qquad \left(\sec(\theta)\right)' = \sec(\theta)\tan(\theta) \qquad \left(\cot(\theta)\right)' = -\csc^2(\theta)$$

Derivatives of exponential and log functions

Remember that exponential and log functions are inverses of one another, *ie*:

$$x = e^{\ln(x)} \quad \text{and} \quad x = \ln(e^x)$$

Exponential and log derivatives – Summary

Exponential: $\left(e^x\right)' = e^x$ and $\left(e^{f(x)}\right)' = e^{f(x)} f'(x)$

Logarithm: $\left(\ln(x)\right)' = \frac{1}{x}$ and $\left(\ln(f(x))\right)' = \frac{1}{f(x)} f'(x)$

For example:

$$f(x) = e^{-0.5x^2} \Rightarrow f'(x) = (e^{-0.5x^2})(-0.5x^2)' = -x\,e^{-0.5x^2}$$

and:

$$f(x) = \ln\left(2x^2+1\right) \Rightarrow f'(x) = \frac{1}{2x^2+1}\left(2x^2+1\right)' = \frac{4x}{2x^2+1}$$

Theory

Higher order derivatives

We can differentiate a function more than once, *eg* we can calculate the derivative of a derivative.

The **n-th order derivative** of the function $y = f(x)$, denoted $f^{(n)}(x)$, is obtained by successively differentiating the function $y = f(x)$ a total of n times.

You should also be aware of the following alternative notation:

- $\dfrac{d^n y}{dx^n}$ or $\dfrac{d^n}{dx^n}f(x)$ is often used as an alternative notation for $f^{(n)}(x)$
- the second and third order derivatives are also often denoted by $f''(x)$ and $f'''(x)$ respectively.

For example, if $y = x^3$, then we have:

$$f'(x) = 3x^2 \quad , \quad f''(x) = \left(f'(x)\right)' = \left(3x^2\right)' = 6x \quad , \quad f'''(x) = \left(f''(x)\right)' = (6x)' = 6$$

In the next calculus lesson, we'll see how these higher order derivatives can be employed in graphing problems, maximum/minimum problems and motion problems.

Implicit differentiation

Consider an equation of the type $g(x,y) = 0$. The plot of all pairs (x,y) satisfying the equation is a more general type of curve than the graph of a function $y = f(x)$.

If we let $g(x,y) = y - f(x)$, then you'll see that the graph of a function is a special case of this more general type of curve.

We want to answer the following question: At the point (x_0, y_0) on the curve $g(x,y) = 0$ (*ie* $g(x_0, y_0) = 0$), what is the slope of the tangent line to the curve (if it exists)?

If the equation $g(x,y) = 0$ can be *explicitly* solved for a relation $y = f(x)$ where $y_0 = f(x_0)$, then the tangent-slope is $f'(x_0)$, if it exists. But if it is not possible to solve $g(x,y) = 0$ for y in terms of x, then we must rely on the technique of **implicit differentiation**.

Implicit differentiation is a two step process.

Step 1: Using derivative rules, differentiate both sides of the equation $g(x,y) = 0$ with respect to x.

Step 2: Make dy/dx the subject of the equation.

Let's look at an example to illustrate this process.

Theory

Consider the circle of radius 5 centered at the origin, $0 = g(x,y) = x^2 + y^2 - 25$, and the point $(3,-4)$ on this circle. How do we calculate the tangent slope at this point? We can do this in two ways.

Method 1: Explicit differentiation

Solving for y in terms of x, we have:

$$y = \pm\sqrt{25 - x^2}$$

The function $y = f_1(x) = \sqrt{25 - x^2}$ is the "upper" semi-circle, and the function $y = f_2(x) = -\sqrt{25 - x^2}$ is the "lower" one.

The point $(3,-4)$ is on the lower semi-circle, so the tangent slope is $f_2'(3)$:

$$\left(-\left(25 - x^2\right)^{1/2}\right)' = \frac{-1}{2}\left(25 - x^2\right)^{-1/2}(-2x) = \frac{x}{\sqrt{25 - x^2}}$$

$$\Rightarrow f_2'(3) = \frac{3}{\sqrt{16}} = 0.75$$

Method 2: Implicit differentiation

Differentiating both sides of $0 = x^2 + y^2 - 25$ with respect to x results in the following:

$$0 = 2x + 2y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \text{tangent slope at } (3,-4) = -\frac{(-3)}{4} = 0.75$$

Derivatives of spliced functions at junction points

Suppose that we have a spliced function defined by the following:

$$y = f(x) = \begin{cases} g(x) & \text{if } x < a \\ c & \text{if } x = a \\ h(x) & \text{if } x > a \end{cases}$$

We saw in Lesson 1 that for this function to be continuous at the junction point $x = a$, the left and right-handed limits had to exist and be equal to the function value:

$$\lim_{x\to a^-} g(x) = \lim_{x\to a^-} f(x) = c = \lim_{x\to a^+} f(x) = \lim_{x\to a^+} h(x)$$

Theory

This condition guarantees that the graphs of $g(x)$ and $h(x)$ meet at the point (a, c). If the functions $g(x)$ and $h(x)$ are both continuous at $x = a$, then the condition above is simply $g(a) = c = h(a)$.

So, suppose that the spliced function is continuous at this junction point.

For the spliced function to be differentiable at this junction point, the two graphs must meet smoothly. To handle this smoothness question, we need to consider **right and left-hand derivatives**:

$$f'_+(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^+} \frac{h(x) - h(a)}{x - a} \qquad \text{right-hand derivative}$$

$$f'_-(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^-} \frac{g(x) - g(a)}{x - a} \qquad \text{left-hand derivative}$$

In general, we have the following theorem:

Theorem

For a spliced function of the form:

$$y = f(x) = \begin{cases} g(x) & \text{if } x < a \\ c & \text{if } x = a \\ h(x) & \text{if } x > a \end{cases}$$

If the spliced function is continuous at $x = a$, then:

$$f'(a) = m \quad \Leftrightarrow \quad g'_-(a) = m = h'_+(a)$$

And if the functions $g(x)$ and $h(x)$ are both differentiable at $x = a$, then:

$$f'(a) = m \quad \Leftrightarrow \quad g'(a) = m = h'(a)$$

These conditions guarantee that the graphs of $g(x)$ and $h(x)$ meet smoothly at the point (a, c).

To see why, let's consider again the function $y = |x|$, which we know is not differentiable at $x = 0$. This function can be viewed as a spliced function:

$$y = f(x) = \begin{cases} g(x) = -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ h(x) = x & \text{if } x > 0 \end{cases}$$

The functions $g(x)$ and $h(x)$ are continuous and differentiable at $x = 0$. Since we have $g(0) = 0 = h(0)$, the spliced function is continuous at the junction point. However, $g'(0) = -1$ and $h'(0) = 1$. So the spliced function is not differentiable at $x = 0$ since the two graphs being spliced do not meet smoothly.

Worked examples

Example 2.1

Reference: BPP

Determine constants c and d so that the following function is differentiable at $x=1$:

$$f(x)=\begin{cases} x^2+x & \text{if } x\le 1 \\ cx+d & \text{if } x>1 \end{cases}$$

(A) $c=1,\quad d=1$

(B) $c=2,\quad d=0$

(C) $c=0,\quad d=2$

(D) $c=3,\quad d=-1$

(E) $c=3,\quad d=1$

Solution

Let $g(x)=x^2+x$ and let $h(x)=cx+d$.

Since they are both polynomial functions, they are differentiable (hence continuous) for all x.

The spliced function $f(x)$ is continuous at the junction point $x=1$ if:

$$g(1)=h(1)$$
$$\Rightarrow 2=c+d$$

Differentiating:

$$g'(x)=2x+1$$
$$h'(x)=c$$

The spliced function is differentiable at $x=1$ if:

$$g'(1)=h'(1)$$
$$\Rightarrow 3=c$$

Substituting $c=3$ into the first of these two relations results in $d=-1$.

So, the correct answer is **D**.

Worked examples

Example 2.2

Reference: BPP

Determine all values of x for which the tangent line to $y = f(x) = x^2 e^{-2x^2}$ is horizontal.

(A) $x = 0$ or $x = \frac{1}{2}$ or $x = -\frac{1}{2}$

(B) $x = \frac{1}{\sqrt{2}}$ or $x = -\frac{1}{\sqrt{2}}$

(C) $x = 0$ or $x = \sqrt{2}$ or $x = -\sqrt{2}$

(D) $x = \ln(2)$ or $x = \ln(-2)$

(E) $x = 0$ or $x = \frac{\sqrt{2}}{2}$ or $x = -\frac{\sqrt{2}}{2}$

Solution

First, we'll calculate the derivative using the chain rule:

$$\left(x^2 e^{-2x^2}\right)' = (2x)\left(e^{-2x^2}\right) + \left(x^2\right)\left(e^{-2x^2}\right)(-4x)$$
$$= e^{-2x^2}\left(2x - 4x^3\right)$$

Since $e^b > 0$ for every number b, the derivative is zero when we have:

$$0 = 2x - 4x^3 = 2x\left(1 - 2x^2\right)$$

The solutions are $x = 0$, $x = \frac{\sqrt{2}}{2}$, and $x = -\frac{\sqrt{2}}{2}$.

So, the correct answer is **E**.

Worked examples

Example 2.3

Reference: BPP

Find the equation of the tangent line to the curve $y^3 x + x^2 y + 2 = 0$ at the point $(1, -1)$ on this curve.

(A) $y = \frac{3x}{4} - \frac{7}{4}$

(B) $y = \frac{3x}{2} - \frac{5}{2}$

(C) $y = \frac{3x}{5} - \frac{2}{5}$

(D) $y = -\frac{3x}{4} + \frac{7}{4}$

(E) $y = -\frac{3x}{2} + \frac{5}{2}$

Solution

It is impossible to explicitly solve this equation for y in terms of x, but we can use implicit differentiation to calculate the derivative.

Differentiating both sides of the equation with respect to x, we have:

$$3y^2 \frac{dy}{dx} x + y^3 + 2xy + x^2 \frac{dy}{dx} = 0$$

Rearranging:

$$\frac{dy}{dx}\left(3y^2 x + x^2\right) = -\left(y^3 + 2xy\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^3 + 2xy}{3y^2 x + x^2}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x,y)=(1,-1)} = -\frac{-1-2}{3+1} = \frac{3}{4}$$

The equation of the line through the point $(x_0, y_0) = (1, -1)$ with slope $m = \frac{3}{4}$ is:

$$y = y_0 + m(x - x_0)$$

$$= -1 + \frac{3}{4}(x - 1)$$

$$= \frac{3x}{4} - \frac{7}{4}$$

So, the correct answer is **A**.

Worked examples

Example 2.4

Reference: BPP

An investment fund of 10,000 at time zero grows to $10{,}000 \times 1.06^t$ in t years.

The force of interest at time t is defined as the ratio of the instantaneous rate of growth of the fund at time t to the amount of the fund at time t.

Calculate the force of interest at time 10 that is earned by this fund.

(A) 0.030
(B) 0.048
(C) 0.058
(D) 0.060
(E) 0.066

Solution

Let $f(t) = 10{,}000 \times 1.06^t$.

Then since:

$$\left(a^x\right)' = a^x \ln(a)$$

we have:

$$f'(t) = 10{,}000 \times 1.06^t \times \ln(1.06)$$

And then the force of interest at time t is:

$$\frac{f'(10)}{f(10)} = \frac{10{,}000 \times 1.06^{10} \times \ln(1.06)}{10{,}000 \times 1.06^{10}} = \ln(1.06) = .0583$$

So, the correct answer is **C**.

Practice questions

Question 2.1

Reference: BPP

For a certain product priced at p per unit sold, $2000-10p$ units will be sold.

Calculate the instantaneous rate of change of revenue with respect to price when the price is 50.

(A) 100
(B) 500
(C) 1000
(D) 1500
(E) 2000

Question 2.2

Reference: BPP

A virus is spreading through a population in a manner that can be modeled by the function

$$g(t)=\frac{A}{1+Be^{-t}}$$

where A is the total population, $g(t)$ is the number infected at time t, and B is a constant.

Determine the rate of infection at time $t=\ln(2)$.

(A) $\dfrac{AB}{4+4B+B^2}$

(B) $\dfrac{2AB}{4+4B+B^2}$

(C) $\dfrac{2AB}{1+B}$

(D) $\dfrac{A}{2+2B+B^2}$

(E) $\dfrac{2A}{2+2B+B^2}$

Practice questions

Question 2.3

Reference: BPP

The total cost of producing n widgets is $C(n) = n^2 + 4n + 100$. The price charged per widget is $P(n) = 100 - n$. Determine the price at which the rate of change of profit with respect to the number of items produced is equal to zero.

(A) 24
(B) 48
(C) 52
(D) 76
(E) 88

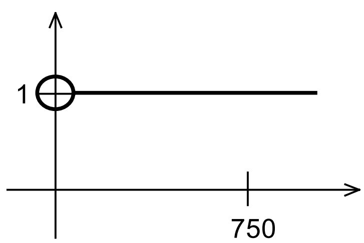
Question 2.4

Reference: May 2000, Question 39

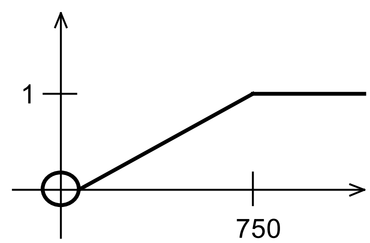
An insurance policy is written that reimburses a policyholder for all losses incurred up to a maximum benefit of 750. Let $f(x)$ be the benefit paid on a loss x.

Which of the following most closely resembles the graph of the derivative of f ?

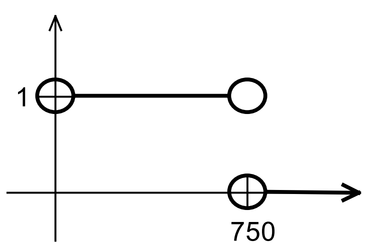
(A)



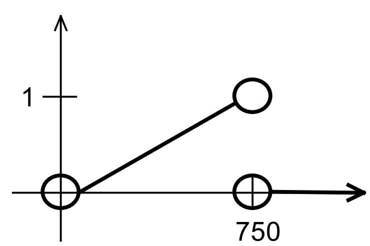
(B)



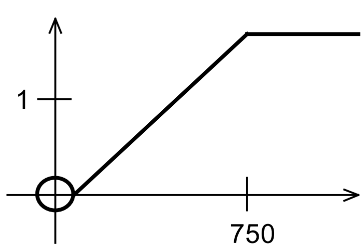
(C)



(D)



(E)



Practice questions

Question 2.5

Reference: BPP

Determine constants c and d so that the following function is differentiable at $x = 1$:

$$f(x) = \begin{cases} x^2 + x & \text{if } x \le 1 \\ \dfrac{cx + d}{x + 1} & \text{if } x > 1 \end{cases}$$

(A) $c = 6.5$ $d = 2.5$

(B) $c = -6.5$ $d = -2.5$

(C) $c = -6.5$ $d = -5.5$

(D) $c = 8$ $d = 4$

(E) $c = 8$ $d = -4$

