Lesson 3

Applications of the Derivative

Overview

In the previous lesson we saw how to calculate the derivative of many types of functions. In this lesson we will study the main applications of derivatives. This area of the syllabus has been tested in a significant number of questions in recent Course 1 exams, so you must be comfortable with the material presented here.

The main applications of derivatives include:

- identifying the shapes and characteristics of graphed functions
- identifying the maximum and minimum values of a function
- calculating rates of change
- calculating limits of functions.

BPP Learning Objectives

This lesson covers the following BPP learning objective:

(C7) *Use differentiation to identify the shape of a graph of a function.*

(C8) *Use differentiation to calculate the maximum or minimum value of a function.*

(C9) *Use L'Hopital's Rule to calculate the limit of a function.*

Theory

Geometric significance of the first derivative

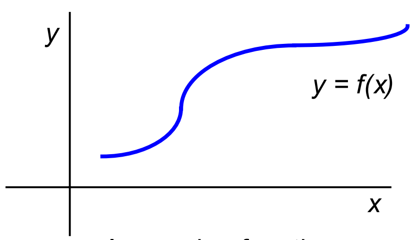
We have already seen that $f'(a)$ can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ on the graph.

This leads to an important use of the first derivative in understanding the shape of a graph and finding the maximum and minimum values of a function. Let's start with a definition.

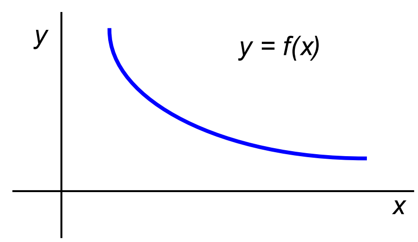
Definition

A function is called **increasing** on the interval $[a,b]$ if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in the interval.

A function is called **decreasing** on the interval $[a,b]$ if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in the interval.



Increasing function



Decreasing function

Notice that the slope of any tangent line to an increasing function is positive, and the slope of any tangent line to a decreasing function is negative. More formally:

Theorem

Suppose that $y = f(x)$ is continuous on $[a,b]$ and differentiable on (a,b).

(i) If $f'(x) > 0$ on the interval (a,b), then $f(x)$ is increasing on $[a,b]$.

(ii) If $f'(x) < 0$ on the interval (a,b), then $f(x)$ is decreasing on $[a,b]$.

Now that we've seen the significance of a positive or negative derivative, let's think about the meaning of a zero derivative.

A horizontal line has a slope equal to zero.

Hence, if $f'(c) = 0$, we know that the tangent line at $(c, f(c))$ is horizontal.

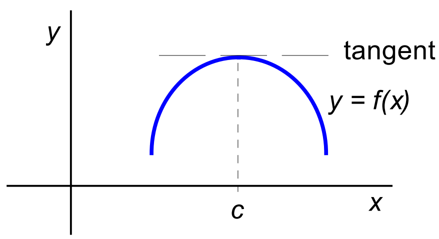
This fact is useful when we're trying to find the local extreme points (*eg* maximum or minimum) of $y = f(x)$.

Theory

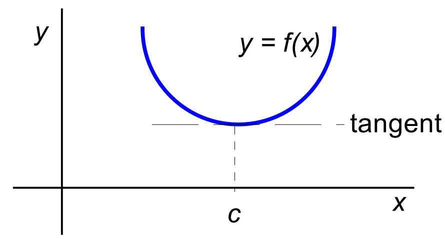
Definition

The point $(c, f(c))$ is called a **local maximum point** if $f(c) \geq f(x)$ for all x in some interval $(c-\delta, c+\delta)$ where $\delta > 0$.

The point $(c, f(c))$ is called a **local minimum point** if $f(c) \leq f(x)$ for all x in some interval $(c-\delta, c+\delta)$ where $\delta > 0$.



Local maximum at (c,f(c))



Local minimum at (c,f(c))

So far so good, but beware of two complications:

- a local extreme point can occur at $x = c$ even if $f'(c)$ does not exist (see Figure 1 below)
- although $f'(c) = 0$, there may not be a local extreme point at $(c, f(c))$ (see Figure 2 below).

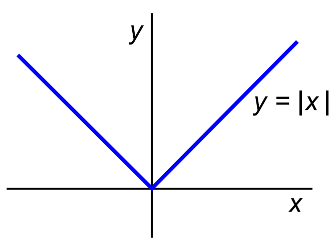


Figure 1: Local minimum but f'(0) undefined

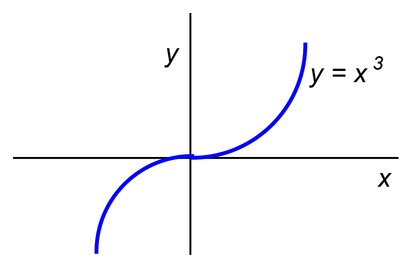


Figure 2: f'(0)=0 but no local extreme point at (0,0)

Definition

A point $(c, f(c))$ is called a **critical point** if either $f'(c) = 0$ or $f'(c)$ is undefined.

The local extreme points occur at critical points, but not every critical point is associated with a local extreme value (as illustrated in Figure 2 above).

Theory

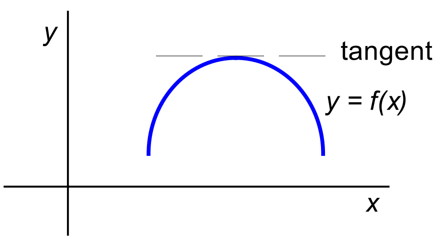
Geometric significance of the second derivative

The second derivative provides information about the **concavity** of the graph $y = f(x)$.

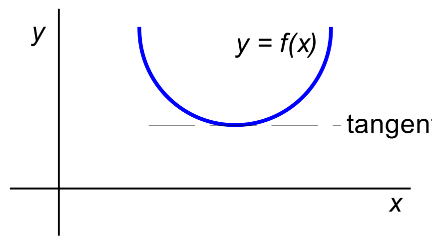
Definition

A function is called **concave down** when the tangent line lies above the graph.

A function is called **concave up** when the tangent line lies below the graph.



Concave down



Concave up

Note: A popular way to remember these definitions is that a graph that is concave down will “spill water” like a cup that is facing down, while a graph that is concave up will “hold water” like a cup that is facing up.

Just as the first derivative describes the rate of change of the function y, the second derivative describes the rate of change of the first derivative, *ie* how the slope of the tangent line changes.

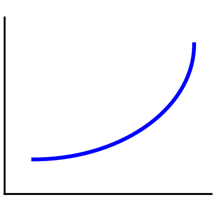
Theorem

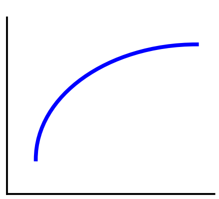
Suppose that $y = f(x)$ is continuous on $[a,b]$ and twice-differentiable on (a,b).

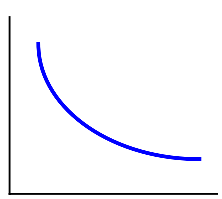
(i) If $f''(x) > 0$ on the interval (a,b), then $f(x)$ is concave up on $[a,b]$.

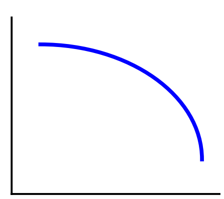
(ii) If $f''(x) < 0$ on the interval (a,b), then $f(x)$ is concave down on $[a,b]$.

So, we can use the first and second derivatives to identify the following shapes of curves.


f(x) increasing
concave up
f' > 0 , f'' > 0


f(x) increasing
concave down
f' > 0 , f'' < 0


f(x) decreasing
concave up
f' < 0 , f'' > 0


f(x) decreasing
concave down
f' < 0 , f'' < 0

Theory

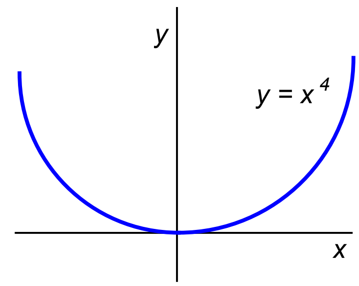
Definition

A point $(c, f(c))$ is called an **inflection point** if the concavity of a graph changes as x moves from just below c to just above c.

Inflection can occur when $f''(c) = 0$ or if $f''(c)$ is undefined. However, $f''(c) = 0$ does not necessarily indicate that inflection occurs at c.

For example, consider the function $y = f(x) = x^4$.

The graph of this function is shown on the right.



The first and second derivatives at $x = 0$ are:

$$f'(0) = 0 \qquad f''(0) = 0$$

but there is no inflection point because the graph is always concave up.

Figure 2 on page 3 of this lesson does illustrate an inflection point. Here, the graph changes at $x = 0$ from concave down (for $x < 0$) to concave up (for $x > 0$)

Optimization problems

In the Course 1 exam, students are often required to calculate the maximum or minimum value of a function on its domain. Questions may be set in a mathematical or a practical context, *eg* calculating the maximum profit that a company can make by adjusting the prices of its products.

We need to be careful to distinguish between a local maximum or minimum, and an absolute maximum or minimum.

Theorem

Suppose that $y = f(x)$ is continuous on $[a, b]$.

Then $f(x)$ has both an absolute maximum value and an absolute minimum value.

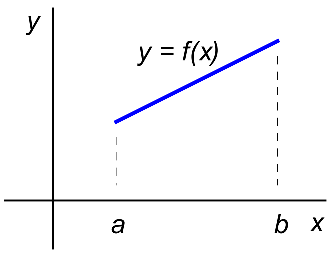
Note that this theorem applies to continuous functions only. To understand why, consider the following function, which has a discontinuity at $x = 0$:

$$y = \begin{cases} 2 & x = 0 \\ \dfrac{1}{x} & 0 < x \le 1 \end{cases}$$

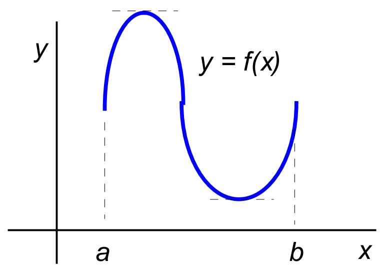
It has a minimum value of 1 at $x = 1$, but it has no maximum. As x approaches zero from the positive side, the expression $y = 1/x$ grows without bound. The range of this function is the interval $[1, \infty)$, which has no largest number.

Theory

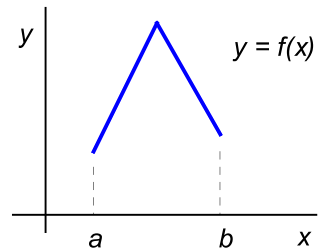
Here are some example graphs showing how absolute extreme values of a continuous function on $[a,b]$ may occur at either an endpoint of the interval or at an interior critical point.



No critical points.
The extreme values
are at the endpoints.



The absolute extreme values
occur at interior critical
points where f'(x) = 0



The absolute maximum occurs at
the interior critical point where
the derivative is undefined.

For example, let's find the maximum and minimum values of $y = x^3 - 9x^2 + 15x + 2$ on the interval $[0,4]$.

The function is a polynomial of degree 3. Polynomials are differentiable (hence continuous) on the whole real line. The first step is to locate any interior critical points.

$$0 = f'(x) = 3x^2 - 18x + 15 = 3(x-1)(x-5) \Rightarrow x = 1 \text{ or } x = 5$$

The only critical point in the interior of the domain $[0,4]$ is at $x = 1$. The second step is to compare function values at the endpoints and interior critical points:

$$f(0) = 2 \qquad f(1) = 9 \qquad f(4) = -18$$

So, the absolute maximum is 9 and it occurs at an interior critical point when $x = 1$. Also, the absolute minimum is -18 and it occurs at the right endpoint.

Let's summarize the derivative criteria for identifying local extreme values. There are two approaches:

Derivative criteria for local extreme values – Summary

Method 1: Using the first and second derivatives

$$f'(a) = 0 \ , \ f''(a) > 0 \text{ (concave up)} \Rightarrow \text{ local minimum at } (a, f(a))$$

$$f'(a) = 0 \ , \ f''(a) < 0 \text{ (concave down)} \Rightarrow \text{ local maximum at } (a, f(a))$$

Method 2: Using the first derivative only

$$f'(a) = 0, f' > 0 \text{ on } (b,a) \text{ and } f' < 0 \text{ on } (a,c) \Rightarrow \text{ local maximum at } (a, f(a))$$

$$f'(a) = 0, f' < 0 \text{ on } (b,a) \text{ and } f' > 0 \text{ on } (a,c) \Rightarrow \text{ local minimum at } (a, f(a))$$

Note: You'll need to use Method 2 if $f'(a) = 0$ and $f''(a) = 0$.

Theory

Related rates problems

Some Course 1 questions involve several related quantities that depend on a variable t that is often thought of as time.

For example, the volume of a sphere of radius r is:

$$V = \frac{4\pi r^3}{3}$$

If a spherical balloon is being inflated with some gas, both the volume and the radius depend on time. Even if we don't have specific formulas for these functions of t, we can still implicitly differentiate V with respect to time in order to find a relation between the rates of change of V and r:

$$\frac{dV}{dt} = \frac{d\left(4\pi r^3/3\right)}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Suppose the volume is increasing at $2ft^3/\text{min}$ when the radius is $5ft$. Then the rate of increase of the radius at this instant in time is:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{1}{4\pi \times 25ft^2} \times 2ft^3/\text{min} = \frac{0.02}{\pi}\,ft/\text{min}$$

Theory

L' Hopital's Rule

Consider a limit problem of the type $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$.

If $L_1 = \lim\limits_{x \to a} f(x)$, $L_2 = \lim\limits_{x \to a} g(x)$, and $L_2 \neq 0$, then the quotient rule applies, *ie*:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

If $L_2 = 0$ and $L_1 \neq 0$, then the left and right hand limits of $f(x)/g(x)$ as x approaches a are both $\pm\infty$.

But what happens if both $L_1 = 0$ and $L_2 = 0$? The limit $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ might not exist, and if it does exist it could be any possible number.

L'Hopital's Rule offers a potentially simpler way to calculate $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$.

L'Hopital's Rule

Suppose that $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$, or that both limits are infinite.

If $f(x)$ and $g(x)$ are differentiable for x near a then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

For example, consider the following limit:

$$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$$

Both the numerator and denominator approach zero. We could factor an $(x-1)$ out of the top and bottom and calculate this limit as follows:

$$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{x^2 + x + 1}{x+1} = \frac{3}{2}$$

However, L'Hopital's Rule offers a quicker solution:

$$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x^3 - 1)'}{(x^2 - 1)'} = \lim_{x \to 1} \frac{3x^2}{2x} = \lim_{x \to 1} \frac{3x}{2} = \frac{3}{2}$$

Theory

So far we have seen how to deal with the limit of a quotient if the quotient rule would result in trying to evaluate the indeterminate expressions $0/0$ or $\pm\infty/\pm\infty$. There is no consistent way to define either of these expressions, or indeed other expressions such as $\infty\times 0$.

L'Hopital's Rule can also be adapted to handle other situations where applying limit rules would result in an indeterminate expression.

For example, consider the following limit problem where one factor approaches infinity and the other approaches zero:

$$\lim_{x\to\infty} x^2 e^{-x}$$

Applying the product rule results in the indeterminate expression $\infty\times 0$ since:

$$\lim_{x\to\infty} x^2 e^{-x} = \left(\lim_{x\to\infty} x^2\right)\left(\lim_{x\to\infty} e^{-x}\right) = \infty\times 0$$

Rewriting the limit as a quotient and applying the quotient rule also results in the indeterminate expression:

$$\lim_{x\to\infty} x^2 e^{-x} \lim_{x\to\infty}\frac{x^2}{e^x} = \frac{\lim_{x\to\infty} x^2}{\lim_{x\to\infty} e^x} = \frac{\infty}{\infty}$$

However, the problem is easily solved using L'Hopital's Rule:

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{\left(x^2\right)'}{\left(e^x\right)'} \qquad \text{using L'Hopital's Rule}$$

$$= \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{(2x)'}{\left(e^x\right)'} \qquad \text{using L'Hopital's Rule again}$$

$$= \lim_{x\to\infty}\frac{2}{e^x} = 0$$

Finally, one other useful result.

For problems of the type $L = \lim_{x\to a} f(x)^{g(x)}$, we have

$$\ln(L) = \lim_{x\to a} g(x)\ln\left(f(x)\right)$$

Let's look at some worked examples.

Worked examples

Example 3.1

Reference: May 2000, Question 16

The total cost, c, to a company for selling n widgets is $c(n) = n^2 + 4n + 100$.

The price per widget is $p(n) = 100 - n$.

What price per widget will yield the maximum profit for the company?

(A) 50
(B) 76
(C) 96
(D) 98
(E) 100

Solution

Profit is the difference between revenue, np, and cost, *ie*:

$$\begin{aligned}\text{Profit} &= \text{Revenue} - \text{Costs}\\ &= np - c(n)\\ &= n(100-n) - \left(n^2 + 4n + 100\right)\\ &= -100 + 96n - 2n^2\end{aligned}$$

Differentiating with respect to n:

$$(\text{Profit})' = 96 - 4n$$

Setting the derivative to zero:

$$0 = (\text{Profit})' = 96 - 4n \quad \Rightarrow \quad n = 24 \quad \Rightarrow \quad p = 100 - n = 76$$

Note that the second derivative is negative, so there is a local maximum at $n = 24$:

$$(\text{Profit})'' = -4$$

Finally:

$$n = 24 \quad \Rightarrow \quad p = 100 - n = 76$$

So, the correct answer is **B**.

Note: If you draw the graph of this function, you'll see that it is a concave down parabola. So the maximum value will occur at the vertex (*ie* the peak above the lone critical point).

Worked examples

Example 3.2

Reference: May 2000, Question 14

A virus is spreading through a population in a manner that can be modeled by the function

$$g(t) = \frac{A}{1+Be^{-t}}$$

where A is the total population, $g(t)$ is the number infected at time t, and B is constant.

What proportion of the population is infected when the virus is spreading the fastest?

(A) 1/3
(B) 1/2
(C) 2/3
(D) 3/4
(E) 1

Solution

The question asks us to calculate $\frac{1}{A}g(t)$ when $g'(t)$ is at its maximum value. So:

$$g'(t) = -A\left(1+Be^{-t}\right)^{-2}\left(1+Be^{-t}\right)' = \frac{ABe^{-t}}{\left(1+Be^{-t}\right)^2}$$

Next, we need to maximize $g'(t)$. We'll do this by rewriting $g'(t)$ in the form:

$$g'(t) = \frac{AB}{e^t\left(1+Be^{-t}\right)^2}$$

Since the numerator is constant, to maximize $g'(t)$ we need to minimize the denominator. Let's calculate the critical points of the denominator by differentiating using the product rule and setting the derivative to zero:

$$0 = \left(e^t\left(1+Be^{-t}\right)^2\right)' = e^t\left(1+Be^{-t}\right)^2 + 2e^t\left(1+Be^{-t}\right)^1\left(-Be^{-t}\right)$$
$$= e^t\left(1+Be^{-t}\right)\left(\left(1+Be^{-t}\right) - 2Be^{-t}\right) = e^t\left(1+Be^{-t}\right)\left(1-Be^{-t}\right)$$

The first two factors on the right side of this final form are always positive. The final factor is zero if we have $Be^{-t} = 1$. This critical point is a local minimum. Finally, plug $Be^{-t} = 1$ into the formula:

$$\frac{g(t)}{A} = \frac{A}{A\left(1+Be^{-t}\right)} = \frac{1}{1+Be^{-t}} = \frac{1}{1+1} = \frac{1}{2}$$

So, the correct answer is **B**.

Worked examples

Example 3.3

Reference: November 2000, Question 13

An actuary believes that the demand for life insurance, L, and the demand for health insurance, H, can be modeled as functions of time, t:

$$L(t) = t^3 + 9t + 100 \qquad 0 \le t \le 4$$

$$H(t) = 6t^2 + 102 \qquad 0 \le t \le 4$$

During the time period $0 \le t \le 4$, the greatest absolute difference between the two demands occurs n times.

Determine n.

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Solution

We need to look at the difference in demands:

$$D(t) = L(t) - H(t) = t^3 - 6t^2 + 9t - 2.$$

The maximum value of $|D(t)|$ can occur at the endpoints $t=0$ or $t=4$, or at a critical point of $D(t)$ between 0 and 4.

Let's begin by finding these critical points by differentiating and setting the derivative equal to zero:

$$0 = D'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3) \quad \Rightarrow \quad t = 1 \text{ or } t = 3$$

Let's check the value of $|D(t)|$ at each of the critical values of t.

$$D(0) = -2 \qquad |D(0)| = 2$$
$$D(1) = 2 \qquad |D(1)| = 2$$
$$D(3) = -2 \qquad |D(3)| = 2$$
$$D(4) = 2 \qquad |D(4)| = 2$$

The greatest absolute difference occurs 4 times.

So, the correct answer is **D**.

Practice questions

Question 3.1

Reference: November 2000, Question 8

An insurance company can sell 20 auto insurance policies per month if it charges 40 per policy. Moreover, for each decrease or increase of 1 in the price, the company can sell 1 more or 1 less policy, respectively. Fixed costs are 100. Variable costs are 32 per policy.

What is the maximum monthly profit that the insurance company can achieve from selling auto insurance policies?

(A) 96
(B) 196
(C) 296
(D) 400
(E) 900

Question 3.2

Reference: November 2000, Question 24

Let f be a function such that $f(x+h)-f(x)=6xh+3h^2$ and $f(1)=5$.

Determine $f(2)-f'(2)$.

(A) 0
(B) 2
(C) 3
(D) 5
(E) 6

Question 3.3

Reference: November 2001, Question 8

A medical researcher conducts a 10-week study of patients infected with a chronic disease. Over the course of the study, the researcher finds that the fraction of patients exhibiting severe symptoms can be modeled as:

$F(t)=te^{-t}$ where t is time elapsed, in weeks, since the study began.

What is the minimum fraction of patients exhibiting severe symptoms between the end of the first week and the end of the seventh week of the study.

(A) 0.0000
(B) 0.0004
(C) 0.0027
(D) 0.0064
(E) 0.3679

Practice questions

Question 3.4

Reference: May 2000, Question 26

The bond yield curve is defined by the function $y(x)$ for $0 \le x \le 30$ where y is the yield on a bond that matures in x years. The bond yield curve is a continuous, increasing function of x and, for any two points on the graph of y, the line segment joining those points lies entirely below the graph of y.

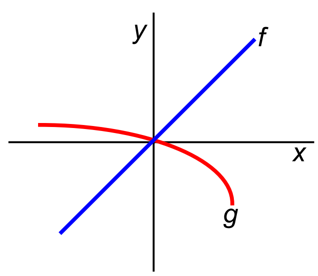
Which of the following functions could represent the bond yield curve?

(A)	$y(x) = a$	a is a positive constant
(B)	$y(x) = a + kx$	a, k are positive constants
(C)	$y(x) = a + k\sqrt{x^3}$	a, k are positive constants
(D)	$y(x) = a + kx^2$	a, k are positive constants
(E)	$y(x) = a + k\ln(x+1)$	a, k are positive constants

Question 3.5

Reference: November 2001, Question 14

The graphs of differentiable functions f and g are shown in the diagram below.



Which of the following is true about $\lim_{x \to 0} \dfrac{f(x)}{g(x)}$?

(A) The limit is less than zero.
(B) The limit is zero.
(C) The limit is 1.
(D) The limit is greater than 1.
(E) The limit does not exist.

Practice questions

Question 3.6 *Reference: November 2001, Question 36*

An insurance company sells health insurance policies to individuals. The company can sell 80 policies per month if it charges 60 per policy. Each increase of 1 in the price per policy the company charges reduces the number of policies the company can sell per month by 1.

Calculate the maximum monthly revenue the company can attain.

(A) 4500
(B) 4800
(C) 4900
(D) 5100
(E) 5200