Lesson 5

Sequences, series and power series

Overview

In this lesson we will introduce the ideas of sequences, series, and power series.

A sequence is an infinite list of numbers, which are linked by a common formula.

A series is a sum of a sequence. We may be interested in the sum of the first *n* terms of a sequence, or the sum of all of the terms (an infinite series). We will study tests that will help us to identify whether the sum of an infinite sequence of numbers converges to some limit, or diverges to infinity.

Power series techniques are a powerful tool in applied mathematics for obtaining approximate solutions to problems that do not have easily obtained exact solutions. We'll identify some important power series approximations of common functions, and we'll study Taylor's Theorem.

BPP Learning Objectives

This lesson covers the following BPP learning objectives:

(C15) *Calculate the limit of a sequence.*

(C16) *Calculate the n-th partial sum of an infinite sequence.*

(C17) *Use convergence tests to determine whether or not an infinite series converges, and if so calculate the limit of the infinite series.*

(C18) *Use power series techniques to make numerical approximations.*

Theory

Sequences

A **sequence** is an infinite list of numbers $\{a_n\}_{n=n_0}^{\infty}$ that has been indexed by the order in which the numbers occur.

For example, the following sequence of numbers:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$$

might be written in the abbreviated algebraic form $\{a_n\}_{n=1}^{\infty}$ where $a_n = \frac{1}{n}$.

The primary concern with any sequence is the behavior of a_n as n gets large. If a_n gets closer to L as n approaches infinity (*ie* as n gets large), then we say that L is the **limit of the sequence** and we write:

$$L = \lim_{n\to\infty} a_n$$

For example, it should be clear that for our previous example:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Theorem

Basic Limits:

(i) $\lim_{n\to\infty} c = c$

(ii) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for any power $p > 0$.

Limit Rules: If $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$ then:

(i) $\lim_{n\to\infty} (a_n \pm b_n) = L_1 \pm L_2$

(ii) $\lim_{n\to\infty} (a_n \times b_n) = L_1 \times L_2$

(iii) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{L_1}{L_2}$ if $L_2 \neq 0$

(iv) $\lim_{n\to\infty} (a_n)^{1/k} = L_1^{1/k}$ for a positive integer k

Link to functions: If $f(x)$ is defined for $x \geq x_0$, and $\lim_{x\to\infty} f(x) = L$, and $a_n = f(n)$ then $\lim_{n\to\infty} a_n = L$

Theory

This final rule is useful because it allows us to use many of the rules that we have learned for calculating the limits of functions, *eg* L'Hopital's Rule.

For example, the limit $\lim\limits_{n\to\infty} \dfrac{n}{\sqrt{1+n^3}}$ can be calculated as $\lim\limits_{n\to\infty} \dfrac{n}{\sqrt{1+n^3}} = \lim\limits_{x\to\infty} f(x)$ where $f(x) = \dfrac{x}{\sqrt{1+x^3}}$.

Using limit rules for limits of functions, we have:

$$\lim_{n\to\infty} \frac{n}{\sqrt{1+n^3}} = \lim_{x\to\infty} \frac{x}{\sqrt{1+x^3}} = \lim_{x\to\infty} \frac{(x)'}{\left(\sqrt{1+x^3}\right)'} = \lim_{x\to\infty} \frac{1}{(3x^2)(1+x^3)^{-1/2}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+x^3}}{3x^2} = \lim_{x\to\infty} \frac{x^{3/2}\sqrt{x^{-3/2}+1}}{3x^2} = \lim_{x\to\infty} \frac{\sqrt{x^{-3/2}+1}}{3\sqrt{x}} = \frac{1}{\infty} = 0$$

Rates of growth

Suppose that $\lim\limits_{n\to\infty} a_n = \infty$ and $\lim\limits_{n\to\infty} b_n = \infty$. The sequence $\{a_n\}_{n=1}^{\infty}$ is said to **go to infinity faster** than the sequence $\{b_n\}_{n=1}^{\infty}$ if $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$.

For example, if $a_n = n^3$ and $b_n = n^2$, then it is clear that $\{a_n\}_{n=1}^{\infty}$ goes to infinity faster than $\{b_n\}_{n=1}^{\infty}$ since $a_n / b_n = n$ still goes to infinity.

The following is a summary of the relative rates of growth of some common functions. These results are useful with limit rules in determining limits of ratios. They will also be helpful with the question of whether or not an infinite series converges to a sum.

Relative Rates of Growth – Summary

(i) $\lim\limits_{n\to\infty} \dfrac{n^p}{n^q} = \infty$ if $p > q$

(ii) $\lim\limits_{n\to\infty} \dfrac{n^p}{\ln(n)} = \infty$ if $p > 0$

(iii) $\lim\limits_{n\to\infty} \dfrac{e^n}{n^p} = \infty$ for any $p > 0$

If $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = 0$ then $\{a_n\}_{n=1}^{\infty}$ is said to **go to zero faster** than $\{b_n\}_{n=1}^{\infty}$ if $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$.

Clearly, $\{a_n\}_{n=1}^{\infty}$ goes to zero faster than $\{b_n\}_{n=1}^{\infty}$ if $\left\{\dfrac{1}{a_n}\right\}_{n=1}^{\infty}$ goes to infinity faster than $\left\{\dfrac{1}{b_n}\right\}_{n=1}^{\infty}$.

Theory

Sequences and series

A **series** is the sum of a sequence of numbers.

Definition

For the infinite sequence $\{a_n\}_{n=1}^{\infty}$, the ***n*-th partial sum** is denoted s_n and is defined by:

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{t=1}^{n} a_t$$

Let's look at two common types of sequences for which there is a general result for the *n*-th partial sum.

In an **arithmetic sequence**, each term is equal to the previous term plus a constant addition (k), *ie*:

$$a_t = a_{t-1} + k$$

So, if the first term is $a_1 = a$ we have:

$$a_1 = a,\ a_2 = a + k,\ a_3 = a + 2k, \cdots,\ a_n = a + (n-1)k$$

The *n*-th partial sum of an arithmetic sequence is given by:

$$s_n = \frac{n}{2}\left(2a + (n-1)k\right) = n\left(a + \frac{1}{2}(n-1)k\right)$$

In a **geometric sequence**, each term is a constant multiple (r) of the previous term, *ie*:

$$a_t = a_{t-1} \times r$$

So, if the first term is $a_1 = a$ we have:

$$a_1 = a,\ a_2 = ar,\ a_3 = ar^2, \cdots,\ a_n = ar^{n-1}$$

The *n*-th partial sum of a geometric sequence is given by:

$$s_n = \frac{a(1-r^n)}{1-r} \qquad \text{for } r \neq 1$$

If $r > 1$, this is often rewritten as:

$$s_n = \frac{a(r^n - 1)}{r - 1}$$

so that the numerator and denominator are both positive.

Note: If $r = 1$ then the *n*-th partial sum for a geometric sequence simplifies to $s_n = a \times n$. This is equivalent to an arithmetic sequence with $k = 0$.

Theory

For example, suppose that a company pays a dividend of 100 in year 1. The dividend grows by 5% (compound) in each subsequent year. To calculate the total dividends paid over the next 20 years, we have:

$$a = 100,\ n = 20,\ r = 1.05$$

$$\Rightarrow s_n = \frac{a(1-r^n)}{1-r} = \frac{100(1-1.05^{20})}{1-1.05} = 3{,}306.60$$

Infinite series

Let's now consider an **infinite series**, *ie* the sum of an infinite sequence of numbers.

Definition

Let $\{s_n\}_{n=1}^{\infty}$ be the sequence of n-partial sums of the infinite sequence $\{a_n\}_{n=1}^{\infty}$.

The **infinite series** $a_0 + a_1 + a_2 + \cdots = \sum_{t=1}^{\infty} a_t$ is said to converge to s if $\lim_{n\to\infty} s_n = s$.

If the sequence of n-partial sums does not converge, the infinite series is said to **diverge**.

Let's look at a few examples.

If $a_1 = 1, a_2 = 1, \cdots, a_t = 1, \cdots$, then $s_n = \sum_{t=1}^{n} 1 = n$. This series diverges to infinity since:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} n = \infty$$

If $a_t = 0.5a_{t-1}$ and $a_1 = 1$ then, using the formula for the n-th partial sum of a geometric sequence, we have:

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \lim_{n\to\infty} \frac{1-0.5^n}{1-0.5} = \frac{1}{0.5} = 2$$

In other words:

$$1 + 0.5 + 0.5^2 + 0.5^3 + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

More generally for a geometric series:

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \quad \text{if } r < 1$$

There is a simple rule to allow us to identify whether or not a geometric series converges or diverges.

Theory

Convergence or divergence of a geometric series

For a geometric sequence with $a_t = a_{t-1} \times r$ and $a_1 = 1$, the infinite series is:

$$1 + r + r^2 + \cdots = \begin{cases} \dfrac{1}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

Testing for convergence or divergence of an infinite series

More generally, how we can tell whether an infinite series converges or diverges? We can break this down into 2 key ideas:

Key idea 1: For an infinite series to have any chance of converging, the terms must approach zero.

This is due to the following elementary idea:

$$s = \lim_{n\to\infty} s_n \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

This leads to the following result:

The *n*-th term test for divergence of a series

If $\lim_{n\to\infty} a_n \neq 0$ or if this limit doesn't exist, then the series $\sum_{t=1}^{\infty} a_t$ cannot converge to a sum.

This idea offers another explanation of why the geometric series diverges unless $|r| < 1$.

For example, if $r = -1.5$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1.5)^n$ does not exist.

Key idea 2: If the terms do not approach zero fast enough, the infinite series will not converge.

In other words, just because the terms of the sequence converge to zero, it is not necessarily true that the series converges to a finite sum.

The classic example is the **harmonic series** $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ where the n-th term is given by $a_n = \frac{1}{n}$.

Grouping the terms, we can see that the infinite series diverges:

$$\begin{aligned} &1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty \end{aligned}$$

Theory

What do these key ideas tell us?

The conclusion is that **an infinite series converges to a finite sum only if its terms converge to zero fast enough**.

This leaves us with a couple of important questions:

- How can we tell if an infinite series converges to a finite sum? In other words, how can we tell if the terms of a sequence converge to zero fast enough?
- If the series does converge, what does it converge to? In many cases, it's not possible to find a simple formula for the infinite sum, so we may need to approximate the sum of a convergent series as the *n*-th partial sum for a large *n*.

We'll start by studying tests for convergence, which will help us to answer the first question.

Convergence tests

Ratio test

Let $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$. Then:

$$\sum_{t=1}^{\infty} a_t \text{ converges if } L<1 \qquad \text{and} \qquad \sum_{t=1}^{\infty} a_t \text{ diverges if } L>1$$

Note: If $L=1$, then this test is inconclusive

Integral test

Let $f(k)=a_k \geq 0$. Then:

$$\sum_{t=1}^{\infty} a_t \text{ converges if } \int_1^{\infty} f(x)dx < \infty \qquad \text{and} \qquad \sum_{t=1}^{\infty} a_t \text{ diverges if } \int_1^{\infty} f(x)dx \text{ is infinite}$$

Alternating series test

Suppose that $a_t \geq 0$, $a_t \geq a_{t+1}$ for all *t*, and $\lim_{t\to\infty} a_t = 0$. Then:

$$\sum_{t=1}^{\infty} (-1)^t a_t \text{ converges}$$

This is known as the **alternating series**.

Furthermore, the absolute difference between the sum s and the *n*-th partial sum is at most a_{n+1}, *ie*:

$$|s-s_n| < a_{n+1}$$

Theory

For example, let's determine whether the series $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges.

Using the Ratio Test, we have:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}/(n+1)!}{2^n/n!}\right| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = 0$$

Since L<1, this series converges.

It's important to note that these 3 tests can help us to decide if a series converges, but they never provide an exact answer as to what the sum is.

The following result is also very useful to know. (It can be proved using the integral test.)

Theorem

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

This result also highlights the idea that the terms of a convergent series must converge to zero fast enough.

The sensitivity of the convergence to the value of p is worthy of emphasis. For the harmonic series, we have $p = 1$ and the series is divergent. But if $p = 1.00001$, say, the series converges.

Theory

Power series

A good way to think of a **power series** is as an infinite degree polynomial.

A typical power series centered at $x = a$ can be written in the form:

$$\sum_{k=0}^{\infty} a_k (x-a)^k$$

It can be shown with the Ratio Test that this series will converge for all x in the interval $(a-R, a+R)$ where R, the **radius of convergence**, is calculated as:

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

It may also converge at the endpoints.

On the interval of convergence, let $s(x)$ denote the sum of the series:

$$s(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

It is not hard to show that the k^{th} derivative of the sum is:

$$s^{(k)}(a) = (k!)a_k$$

Taylor's theorem is a device for starting with a function $f(x)$ and writing it as a convergent power series.

Taylor's Theorem

Suppose that the function $f(x)$ has derivatives of all orders at $x = a$. Then for any positive integer n it is possible to express the function as

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{\substack{P_n(x) \text{ an n-th degree} \\ \text{polynomial approximation}}} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\substack{R_n(x) \text{ the remainder or} \\ \text{error in the approximation}}}$$

where the number c is between x and a.

For any x such that the remainder term approaches zero as n goes to infinity, the function $f(x)$ can be written as a convergent power series. These ideas are used to express a number of important functions as power series. For such functions, note that the n^{th} partial sum, $P_n(x)$, would serve as a good approximation to $f(x)$ for sufficiently large n. For a specified level of accuracy, ε, the remainder term formula above can be used to find the number of terms needed in the partial sum such that $|f(x) - P_n(x)| = |R_n(x)| < \varepsilon$ for all x in some interval centered at $x = a$.

Theory

Important power series

(i) $$\frac{1}{1-x}=1+x+x^2+\cdots=\sum_{k=0}^{\infty}x^k \quad \text{if } -1<x<1$$

(ii) $$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=\sum_{k=0}^{\infty}\frac{x^k}{k!} \quad \text{for all } x$$

(iii) $$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} \quad \text{if } -1<x<1$$

(iv) $$\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} \quad \text{for all } x$$

$$\cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} \quad \text{for all } x$$

Power series techniques

There are some useful techniques for working with power series.

Substitution

We can find the power series for a function by substituting the appropriate terms in an existing power function.

For example, we can convert the power series e^x to a power series for the function $e^{-0.5x^2}$ by replacing all x's on both sides of the formula with $-0.5x^2$.

Hence:

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$\Rightarrow e^{-0.5x^2}=1+\left(-0.5x^2\right)+\frac{\left(-0.5x^2\right)^2}{2!}+\frac{\left(-0.5x^2\right)^3}{3!}+\cdots=1-\frac{x^2}{2}+\frac{x^4}{8}-\frac{x^6}{48}+\cdots$$

Integration

A convergent power series can be integrated term by term on its interval of convergence to produce another convergent power series with the same interval of convergence.

For example, consider the standard normal distribution probability density function:

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-0.5x^2}$$

Theory

The probability that a random variable with a standard normal distribution is less that or equal to x is the cumulative distribution function $\Phi(x)=\int_{-\infty}^{x} f(t)\,dt$.

From our previous example, we know that:

$$e^{-0.5x^2}=1+\left(-0.5x^2\right)+\frac{\left(-0.5x^2\right)^2}{2!}+\frac{\left(-0.5x^2\right)^3}{3!}+\cdots=1-\frac{x^2}{2}+\frac{x^4}{8}-\frac{x^6}{48}+\cdots$$

A power series expansion for this function can be obtained by integration.

If $x \geq 0$ we have:

$$\Phi(x)=0.5+\int_0^x f(t)\,dt$$

since one half of the probability is to the left of zero.

As a result, we have the following:

$$\Phi(x)=0.5+\frac{1}{\sqrt{2\pi}}\int_0^x 1-\frac{t^2}{2}+\frac{t^4}{8}-\frac{t^6}{48}+\cdots dt=0.5+\frac{1}{\sqrt{2\pi}}\left(x-\frac{x^3}{6}+\frac{x^5}{40}-\frac{x^7}{336}+\cdots\right)$$

Using the first four terms in this series results in the approximation:

$$\Phi(1)=0.5+\frac{1}{\sqrt{2\pi}}\left(1-\frac{1}{6}+\frac{1}{40}-\frac{1}{336}+\cdots\right)=0.5+\frac{0.85536}{\sqrt{2\pi}}=0.84124\ .$$

This is close to the value of $\Phi(1)=0.8413$ given in the standard normal probability table. The approximation gets closer if we use more terms.

Differentiation

A convergent power series can be differentiated term by term on its interval of convergence to produce another convergent power series with the same interval of convergence.

For example, from the geometric power series:

$$\frac{1}{1-x}=1+x+x^2+\cdots=\sum_{k=0}^{\infty}x^k \quad \text{if } -1<x<1$$

we can differentiate both sides to obtain the following convergent power series:

$$\frac{1}{(1-x)^2}=1+2x+3x^2+\cdots=\sum_{k=1}^{\infty}kx^{k-1}\ ,\quad |x|<1$$

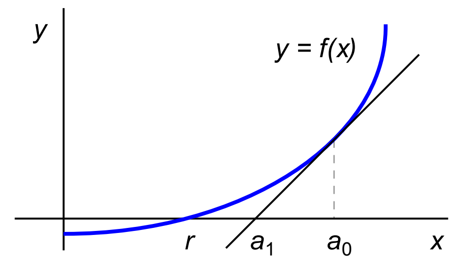
We'll conclude this lesson by looking at one other use of sequences.

Theory

Newton's method for solving equations

Suppose we desire to find a root r of the equation $f(x)=0$. Newton's method provides a recursively-generated sequence of approximations that converges to the root r provided that a good first guess a_0 is available.

The next guess, a_1, is obtained as the value of x where the tangent line to $y=f(x)$ at the point $(a_0, f(a_0))$ crosses the x-axis.



The equation of the tangent line is:

$$y = f(a_0) + f'(a_0)(x - a_0)$$

So, we can find the value of a_1 as:

$$0 = f(a_0) + f'(a_0)(a_1 - a_0) \quad \Rightarrow \quad a_1 = a_0 - \frac{f(a_0)}{f'(a_0)}$$

From the picture, you can see that a_1 is nearer to the root r than our original guess a_0.

This process is repeated so that we have a recursively-generated sequence $\{a_n\}$ beginning with the first guess a_0 and continuing according to the recursive relation:

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} \quad n = 0,1,2,\cdots$$

You can see from the picture that $\lim_{n\to\infty} a_n = r$.

In practice, we'll stop after n iterations when $f(a_{n+1})$ is reasonably close to zero, giving a_{n+1} as our approximation to r. Convergence is usually rapid.

For example, we can use Newton's method to approximate $\sqrt[3]{2}$.

This number is a root of the equation $f(x) = x^3 - 2 = 0$, so we have (setting $a_0 = 1$):

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} = a_n - \frac{a_n^3 - 2}{3a_n^2}$$

$$\Rightarrow a_1 = a_0 - \frac{a_0^3 - 2}{3a_0^2} = 1.3333, \quad a_2 = a_1 - \frac{a_1^3 - 2}{3a_1^2} = 1.2639, \quad a_3 = a_2 - \frac{a_2^3 - 2}{3a_2^2} = 1.2599$$

Valuing the function at this estimate:

$$f(1.2599) = 0.00059 \quad \Rightarrow \quad \sqrt[3]{2} \approx 1.2599$$

Worked examples

Example 5.1

Reference: BPP

Let $a = \lim_{n\to\infty} \frac{1000^n}{n!}$ $\quad b = \lim_{n\to\infty} \frac{(\ln(n))^2}{n}$ $\quad c = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n$

Calculate the values of a, b, and c.

(A) $a=\infty,\ b=\infty,\ c=ae^a$

(B) $a=0,\ b=\infty,\ c=e^a$

(C) $a=\infty,\ b=0,\ c=\ln(a)$

(D) $a=0,\ b=0,\ c=e^a$

(E) $a=0,\ b=0,\ c=ae^a$

Solution

To calculate a, consider the ratio of consecutive terms:

$$\frac{a_{n+1}}{a_n} = \frac{1000^{n+1}}{(n+1)!}\times\frac{n!}{1000^n} = \frac{1000}{n+1} < 1 \quad \text{for } n > 1000$$

So, although the sequence increases initially, it decreases for $n > 1000$. Hence, $a = \lim_{n\to\infty}\frac{1000^n}{n!} = 0$

To calculate b:

$$\lim_{n\to\infty}\frac{(\ln(n))^2}{n} = \lim_{x\to\infty}\frac{(\ln(x))^2}{x} = \lim_{x\to\infty}\frac{2\ln(x)/x}{1} \quad \text{(L'Hopital's Rule)}$$

$$= \lim_{x\to\infty}\frac{2\ln(x)}{x} = \lim_{x\to\infty}\frac{2/x}{1} = \lim_{x\to\infty}\frac{2}{x} = 0 \quad \text{(L'Hopital's Rule again)}$$

To calculate c:

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = L \quad \Rightarrow \quad \ln(L) = \lim_{n\to\infty}\ln\left(\left(1+\frac{a}{n}\right)^n\right) = \lim_{n\to\infty} n\ln\left(1+\frac{a}{n}\right)$$

$$\Rightarrow \ln(L) = \lim_{n\to\infty}\frac{\ln(1+a/n)}{1/n} = \lim_{x\to\infty}\frac{\ln(1+a/x)}{1/x} = \lim_{x\to\infty}\frac{(1+a/x)^{-1}(-a/x^2)}{-1/x^2} \quad \text{(L'Hopital's Rule)}$$

$$= \lim_{x\to\infty}\frac{a}{1+a/x} = \frac{a}{1} = a$$

$$\Rightarrow L = e^{\ln(L)} = e^a$$

So, the correct answer is **D**.

Worked examples

Example 5.2

Reference: BPP

Calculate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Hint: Use the identity $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

(A) 3/4
(B) 1
(C) 3/2
(D) 2
(E) The sum does not exist.

Solution

We'll start by finding a formula for the *n*-th partial sum of the series.

From the identity in the hint, it follows that:

$$\begin{aligned} s_n &= \sum_{t=1}^{n} \frac{1}{t(t+1)} \\ &= \sum_{t=1}^{n} \left(\frac{1}{t} - \frac{1}{t+1} \right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

So, we can find the limit of the infinite series as follows:

$$s = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1} \right) = 1 - 0 = 1$$

So, the correct answer is **B**.

Note: Examples of this type are the exception, rather than the rule. It is quite unusual to be able to find such a compact formula for the *n*-th partial sum.

Worked examples

Example 5.3

Reference: BPP

Which of the following infinite series converge?

I. $\displaystyle\sum_{n=0}^{\infty} \frac{n^4}{2^n}$

II. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{n}{1000n+1}$

III. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\left(n^2+1\right)^2}$

(A) I only
(B) III only
(C) I and II only
(D) I and III only
(E) I, II, and III

Solution

I. Using the ratio test:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^4}{2^{n+1}}\cdot\frac{2^n}{n^4}\right| = \frac{1}{2}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^4 = \frac{1}{2}\left(\lim_{n\to\infty}\frac{n+1}{n}\right)^4$$

$$= \frac{1}{2}\times 1^4 = \frac{1}{2} < 1 \quad \Rightarrow \quad \text{The series converges since } L < 1$$

II. This series diverges due to the n-th term test. The n-th term is $a_n = \dfrac{\pm n}{1000n+1}$.

For large n, this is alternately near -0.001 and 0.001.

Hence, $\lim\limits_{n\to\infty} a_n$ does not exist, and the series cannot converge.

III. Note that $a_n = \dfrac{n}{n^4+2n^2+1} < \dfrac{n}{n^4} = \dfrac{1}{n^3} = b_n$ and that $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^3}$ converges.

For a series with positive terms the partial sums are either bounded (*ie* finite) and converge to a sum, or else they are unbounded and diverge to infinity. Since the infinite series of $\{b_n\}$ converges, its partial sums are bounded. Hence so are the partial sums of the smaller terms given by $\{a_n\}$.

So the infinite series of $\{a_n\}$ must converge to a finite sum as well.

Note: This technique is called a **comparison test**.

So, the correct answer is **D**.

Practice questions

Question 5.1

Reference: November 2000, Question 29

Insurance losses are not always reported in the year they occur. In fact, some losses are reported many years later. The year in which a loss occurs is called the occurrence year.

For a given year, let R_n denote the total number of losses reported in the occurrence year and the following n years. An actuary determines that R_n can be modeled by the sequence:

$$R_{n+1} = 2^{0.75^n} R_n \qquad \text{for } n = 0, 1, 2, \cdots$$

For the occurrence year 1999, 250 losses were reported during 1999. In other words, $R_0 = 250$.

How many more occurrence year 1999 losses does the model predict will be reported in years subsequent to 1999?

(A) 1750
(B) 2000
(C) 3172
(D) 3422
(E) 3750

Question 5.2

Reference: November 2000, Question 39

In a certain town, the number of deaths in year t due to a particular disease is modeled by:

$$\frac{90{,}000}{(t+3)^3} \qquad \text{for } t = 1, 2, 3, \cdots$$

Let N be the total number of deaths that the model predicts will occur in the town after the end of the 27th year.

Which of the following intervals contains N?

(A) $39.5 \le N < 43.0$
(B) $43.0 \le N < 46.5$
(C) $46.5 \le N < 50.0$
(D) $50.0 \le N < 53.5$
(E) $53.5 \le N < 57.0$

Practice questions

Question 5.3

Reference: November 2001, Question 10

Let $\{a_n\}$ be a sequence of real numbers.

For which of the following does the infinite series $\sum_{n=1}^{\infty}\left(a_n+\frac{1}{n}\right)$ converge?

(A) $a_n = 1$

(B) $a_n = \frac{1}{n}$

(C) $a_n = \frac{1}{n^2}$

(D) $a_n = \frac{(-1)^n}{n}$

(E) $a_n = \frac{1-n}{n^2}$

Question 5.4

Reference: May 2000, Question 29

An investor buys on share of stock in an internet company for 100. During the first 4 days that he owns the stock, the share price changes as follows (measured relative to the prior day's price):

Day 1	Day 2	Day 3	Day 4
up 30%	down 15%	unchanged	down 10%

If the pattern of relative price movements observed on the first four days is repeated indefinitely, how will the price of a share of stock behave in the long run?

(A) It converges to 0.00.
(B) It converges to 99.45.
(C) It converges to 101.25.
(D) It oscillates between two finite values without converging.
(E) It diverges to infinity.

Practice questions

Question 5.5

Reference: May 2001, Question 2

A stock pays annual dividends. The first dividend is 8 and each dividend thereafter is 7% larger than the previous dividend.

Let m be the number of dividends paid by the stock when the cumulative amount paid first exceeds 500.

Calculate m.

(A) 23
(B) 24
(C) 25
(D) 26
(E) 27

Question 5.6

Reference: Sample exam, Question 12

An investor invests 100. The value, I, of the investment at the end of one year is given by the equation

$$I = 100\left(1 + \frac{c}{n}\right)^n$$

where c is the nominal annual rate of interest and n is the number of interest compounding periods in one year.

Determine I if there are an infinite number of compounding periods in one year.

(A) 100
(B) $100\,e\,c$
(C) $100\,e^{c}$
(D) $100\,e^{1/c}$
(E) ∞