Course 1 Key Concepts
Calculus

Lesson 6

Parametric equations and polar coordinates

Overview

In this lesson we will introduce the ideas of parametric equations and polar coordinates in the plane $\mathbb{R}^2$.

Parametric equations are used to create a mathematical model of the motion over time of a particle along a curved path. We will see how to calculate the speed of this motion as well as the distance traveled along the curved path.

Polar coordinates offer another way to locate points in the plane. Polar coordinates are of the form (r,θ), where the number *r* represents the distance of the point from the origin, and θ measures the angle of the location of the point relative to the origin. As we'll see, some problems that are difficult to handle in the more common rectangular coordinate system are a bit simpler when converted to polar coordinates.

BPP Learning Objectives

This lesson covers the following BPP learning objectives:

(C19) Use parametric equations to determine the path of a particle, to compute its speed and velocity vector, and to determine distance traveled along the path.

(C20) Use the relations between rectangular and polar coordinates to convert equations in one system to equivalent equations in the other system.

(C21) Calculate the area enclosed by a polar curve and the length of a polar curve.

Theory

Vectors in the plane

In order to study parametric equations we must first introduce the idea of vectors in the plane.

A vector has two attributes: **direction** and **length** (or **magnitude**).

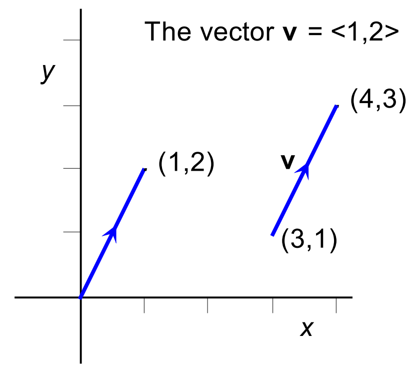
Think of a vector as an "arrow" in the plane.

A vector in the plane will be written in the form:

$$\mathbf{v} = \langle x, y \rangle$$

The real numbers x and y are referred to as the **components** of the vector.

A given vector can be represented pictorially by placing the "tail" at the point $P_0 = (x_0, y_0)$ and its "tip" at the point $P_1 = (x_1, y_1)$ where:



$$x_1 = x_0 + x, \quad y_1 = y_0 + y$$

This type of vector is referred to as the **displacement vector** from P_0 to P_1 and is denoted by $\overrightarrow{P_0 P_1}$.

When $P_0 = (0,0)$ and $P_1 = (x, y)$ it is referred to as the **position vector** corresponding to the point $P = (x, y)$.

Several important arithmetic operations are performed on vectors:

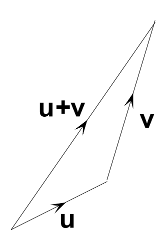
Vectors – Arithmetic Operations

Vector addition/subtraction:	$\langle x_1, y_1 \rangle \pm \langle x_2, y_2 \rangle = \langle x_1 \pm x_2, y_1 \pm y_2 \rangle$
Scalar multiplication:	$r\langle x, y \rangle = \langle rx, ry \rangle$
Magnitude:	$\lvert \langle x, y \rangle \rvert = \sqrt{x^2 + y^2}$

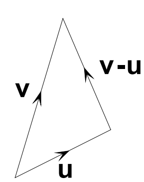
Addition can be viewed as a "tip to tail" picture.

Scalar multiplication can be viewed as stretching ($|r| > 1$) or shrinking ($|r| < 1$) along with the possibility of reversing ($r < 0$).

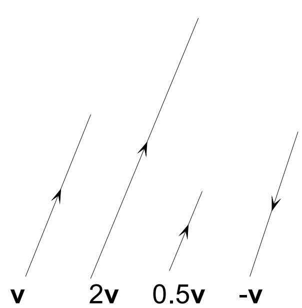
The magnitude of the vector $\mathbf{v} = \langle x, y \rangle$ is the distance from the origin to the point (x, y).



Addition



Subtraction



Scalar Multiplication

Theory

Vector valued functions of a real variable

For each real number t in the interval $[a,b]$ a **vector valued function** assigns a vector $\mathbf{v}(t)=\langle f(t),g(t)\rangle$ whose components are real valued functions.

For these functions we need to consider limits, continuity, derivatives, and anti-derivatives:

Definition

Limits: $\lim\limits_{t\to t_0}\langle f(t),g(t)\rangle=\langle L_1,L_2\rangle$ if $|\langle f(t),g(t)\rangle-\langle L_1,L_2\rangle|\to 0$ as $t\to t_0$

Continuity: $\mathbf{v}(t)=\langle f(t),g(t)\rangle$ is said to be continuous at t_0 if $\lim\limits_{t\to t_0}\mathbf{v}(t)=\mathbf{v}(t_0)$

Derivative: $\mathbf{v}'(t_0)=\lim\limits_{t\to t_0}\dfrac{\mathbf{v}(t)-\mathbf{v}(t_0)}{t-t_0}$ if the limit exists.

Note: Dividing a vector by $r\in\mathbb{R}$ is the same as scalar multiplication by the number $1/r$.

Anti-derivative: An anti-derivative of the vector valued function $\mathbf{v}(t)=\langle f(t),g(t)\rangle$ is a vector valued function $\mathbf{w}(t)$ such that $\mathbf{w}'(t)=\mathbf{v}(t)$.

We also have the following theorem, which effectively states that calculus of vector valued functions boils down to calculus for the two component functions.

Theorem

(i) $\lim\limits_{t\to t_0}\langle f(t),g(t)\rangle=\langle L_1,L_1\rangle \;\Leftrightarrow\; \lim\limits_{t\to t_0}f(t)=L_1$ and $\lim\limits_{t\to t_0}g(t)=L_2$

(ii) $\mathbf{v}(t)=\langle f(t),g(t)\rangle$ is continuous at t_0 $\Leftrightarrow$ both $f(t)$ and $g(t)$ are continuous at t_0

(iii) $\mathbf{v}'(t_0)=\langle f'(t_0),g'(t_0)\rangle \;\Leftrightarrow\; f'(t_0)$ and $g'(t_0)$ both exist

(iv) $\mathbf{w}(t)=\langle F(t),G(t)\rangle$ is an anti-derivative of $\mathbf{v}(t)=\langle f(t),g(t)\rangle \;\Leftrightarrow\; F'(t)=f(t)$ and $G'(t)=g(t)$

For example:

$$\lim_{t\to1}\left\langle\frac{t^2-1}{t-1},e^{-t}\right\rangle=\left\langle\lim_{t\to1}\frac{t^2-1}{t-1},\ \lim_{t\to1}e^{-t}\right\rangle=\left\langle\ \underbrace{\lim_{t\to1}\frac{2t}{1}}_{\substack{\text{L'Hopital's}\\ \text{Rule}}}\ ,\ \underbrace{e^{-1}}_{\substack{\text{continuity}\\ \text{of } g(t)=e^{-t}}}\ \right\rangle=\langle 2,e^{-1}\rangle$$

Theory

As a further example, suppose that $\mathbf{v}(t) = \langle f(t), g(t) \rangle = \langle 2t+1, t^2 - 2t \rangle$.

Since both component functions are continuous and differentiable, it follows that $\mathbf{v}(t)$ is continuous and:

$$\mathbf{v}'(t) = \langle f'(t), g'(t) \rangle = \langle (2t+1)', (t^2 - 2t)' \rangle = \langle 2, 2t-2 \rangle .$$

Also, an anti-derivative of $\mathbf{v}(t)$ is given by

$$\mathbf{w}(t) = \int \mathbf{v}(t)\,dt = \langle \int f(t)\,dt , \int g(t)\,dt \rangle$$
$$= \langle \int 2t+1\,dt , \int t^2 - 2t\,dt \rangle = \langle t^2 + t + c, \frac{1}{3}t^3 - t^2 + d \rangle$$

Parametric curves in the plane

The motion of a "particle" (or a point) along a curved path in $\mathbb{R}^2$ over the time period $a \le t \le b$ can be described by letting $x = f(t), y = g(t)$ and then plotting the points $(f(t), g(t))$ for all $t \in [a,b]$.

Here, t is called the **time parameter**.

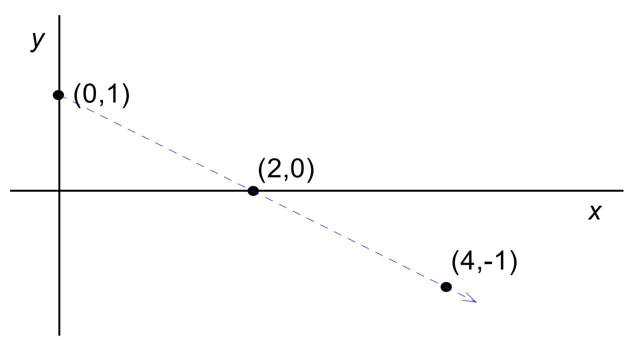
The point $(x,y) = (f(t), g(t))$ is referred to as the **location at time *t*** and the equations $x = f(t)$ and $y = g(t)$ are referred to as the **parametric equations** of the motion.

Vector calculus proves to be very useful in the analysis of a parametric motion. For example, suppose that:

$$x = f(t) = 2t \text{ and } y = g(t) = 1 - t \text{ for } 0 \le t \le 2 .$$

We can calculate the location at times 0, 1 and 2:

At time $t = 0$: $x = 0$ $y = 1$
At time $t = 1$: $x = 2$ $y = 0$
At time $t = 2$: $x = 4$ $y = -1$



We can get an idea of the shape of the path by plotting these locations.

It appears that the points lie along a line. How can we tell if this is true?

A common technique used for this purpose is called **elimination of the parameter**.

One of the two parametric equations is solved for t and then it is substituted into the other equation. Here the first equation yields:

$$t = 0.5x$$

If this is substituted into the second equation, then the result is the equation:

$$y = 1 - 0.5x$$

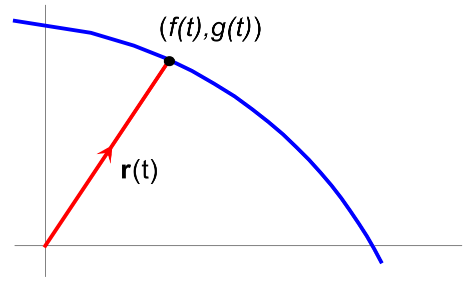
This is the formula for the dotted line in the figure.

Theory

Suppose that the location of a particle at time t is given by the parametric equations $x = f(t), y = g(t)$.

The **position vector** $\mathbf{r}(t) = \langle f(t), g(t) \rangle$ is a displacement vector from the origin to the location at time t.

Now we are ready to analyze the **velocity vector** $\mathbf{v}(t) = \mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$ of the motion and the speed of the particle at any point in time.



The velocity vector is quite easily computed from its definition as the derivative of the position vector, but let's consider its geometric properties:

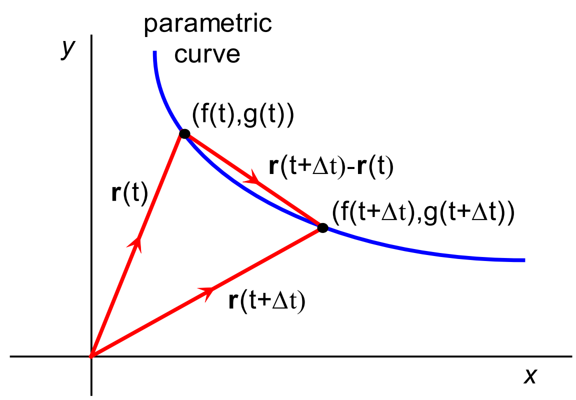
- If the velocity vector is plotted as a displacement vector with its tail at the point $(x, y) = (f(t), g(t))$ (*ie* the location at time t), then it overlies the tangent line to the curve at this point and it points in the direction of motion.
- The length of the velocity vector is the speed at which the particle is travelling along the curved path.

Consider the figure below and recall the limit definition of the derivative of a vector valued function:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

From the figure it is easily seen that the direction of the vector $\mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ approaches the tangent direction of the parametric curve at the point $(f(t), g(t))$ as $\Delta t \to 0$.

Hence the velocity vector overlies the tangent line to the curve at this same point.



In order to consider speed, let $s = s(t)$ denote the distance traveled along the curved path from time a until time t. Speed is rate of change of distance with respect to time, that is $s'(t)$. In order to gain some insight as to why the length of the velocity vector is equal to the speed of the motion consider again the figure above.

The change in $s(t)$ along the curved path from time t to time $t + \Delta t$ is approximately the same as the straight-line distance between the two locations.

The straight-line distance is in turn the length of the displacement vector $|\mathbf{r}(t + \Delta t) - \mathbf{r}(t)|$:

$$\frac{s(t + \Delta t) - s(t)}{\Delta t} = \frac{\text{curved distance}}{\Delta t} \approx \frac{\text{straight-line distance}}{\Delta t} = \frac{|\mathbf{r}(t + \Delta t) - \mathbf{r}(t)|}{\Delta t}$$

$$\Rightarrow s'(t) = |\mathbf{r}'(t)| = |\mathbf{v}(t)|$$

Theory

Here is a summary of important calculations that can be made from the parametric equations $x=f(t), y=g(t)$ of a motion along a curved path:

Parametric Motion – Summary

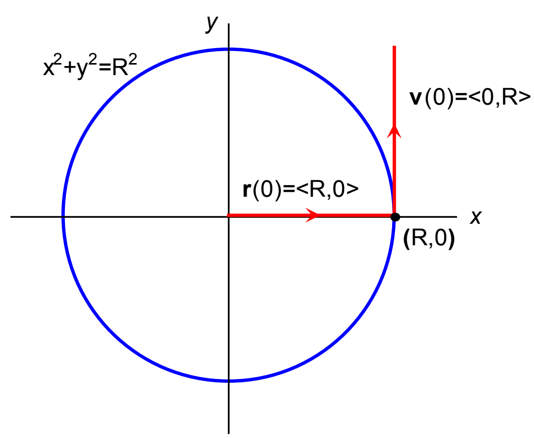
Velocity vector	$\mathbf{v}(t)=\mathbf{r}'(t)=\langle f'(t),g'(t)\rangle$
Speed	$s'(t)=\frac{ds}{dt}=\lvert\mathbf{v}(t)\rvert=\lvert\langle f'(t),g'(t)\rangle\rvert=\sqrt{(f'(t))^2+(g'(t))^2}$
Distance	$s(t)=\int_a^t s'(u)\,du=\int_a^t\sqrt{(f'(u))^2+(g'(u))^2}\,du$
Tangent slope	$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{g'(t)}{f'(t)}$
Position vector	$\mathbf{r}(t)=\int\mathbf{r}'(t)dt=\int\mathbf{v}(t)dt=\int\langle f'(t),g'(t)\rangle dt=\left\langle\int f'(t)dt,\int g'(t)dt\right\rangle$

For example, consider the parametric equations:

$x=f(t)=R\cos(t)$ and $y=g(t)=R\sin(t)$ where $0\le t\le 2\pi$

Note that $x^2+y^2=R^2\cos^2(t)+R^2\sin^2(t)=R^2$, so we can see that the path must trace a circle or radius R centered at the origin (see the diagram below).

By computing and plotting the position and velocity vectors at several points in time, we can see that the motion starts in a counterclockwise direction at the point $(R,0)$ at time 0 and returns to this point for the first time at time $t=2\pi$.



We also have:

$$\mathbf{v}(t)=\langle R\cos(t),R\sin(t)\rangle'=\langle R\cos'(t),R\sin'(t)\rangle$$
$$=\langle -R\sin(t),R\cos(t)\rangle$$
$$\mathbf{v}(0)=\langle 0,R\rangle$$

$$s'(t)=\frac{ds}{dt}=\lvert\mathbf{v}(t)\rvert=\sqrt{(-R\sin(t))^2+(R\cos(t))^2}=\sqrt{R^2}=R$$

And the total distance traveled is:

$s(2\pi)=\int_0^{2\pi}s'(t)\,dt=\int_0^{2\pi}R\,dt=2\pi R$ (which you should recognize as the circumference of the circle)

Theory

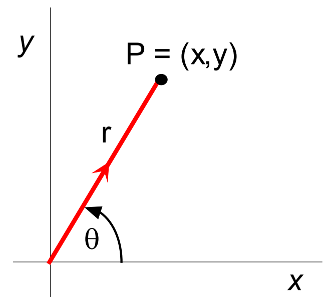
Polar coordinates in the plane

Polar coordinates provide an alternative method to the usual rectangular coordinate system for locating points in the plane.

In the figure on the right, the point P has rectangular coordinates (x, y) and polar coordinates (r, θ).

The number r represents the distance from P to the origin.

The angle θ is measured in a counterclockwise direction from the positive direction of the x-axis.



One unfortunate and potentially confusing aspect of polar coordinates is that a single point P has infinitely many polar representations in addition to the standard one in the picture above where $r > 0$ and $0 \le \theta < 2\pi$.

When n is an integer, the polar coordinates r and $\theta + 2n\pi$ will also describe the point P since there are 2π radians in a 360 degree angle.

Also, in order to deal with graphing a polar curve (*ie* plotting all points whose polar coordinates satisfy an equation of the type $r = f(\theta)$) the number r is allowed to be negative.

Polar coordinates of the point P can also be given in the form $-r$ and $\theta + (2n+1)\pi$. If $r < 0$, we first determine the *ray* that is $\theta + (2n+1)\pi$ radians in a counterclockwise direction from the positive X axis and then we plot a point $|r|$ units in a direction opposite to the direction of this ray.

Let's summarize the main relations between rectangular and polar coordinates.

Relations between rectangular and polar coordinates

$x = r\cos(\theta)$	$r^2 = x^2 + y^2$	$\cos(\theta) = x / \sqrt{x^2 + y^2}$
$y = r\sin(\theta)$	$\tan(\theta) = y / x$	$\sin(\theta) = y / \sqrt{x^2 + y^2}$

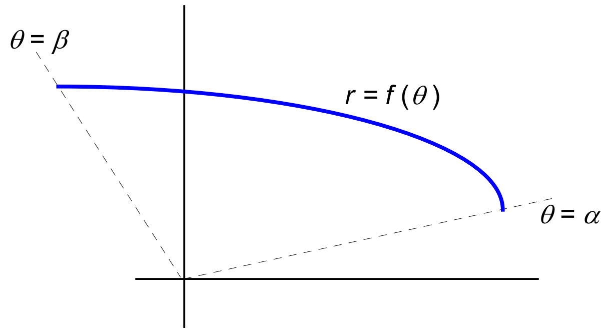
Possible polar coordinates of the point *P* = (*x* , *y*)

$r, \theta + 2n\pi$ $\qquad$ $-r, \theta + (2n+1)\pi$

Theory

Polar curves in the plane

A **polar curve** consists of all points whose polar coordinates are r, θ where $r = f(\theta)$ and $\alpha \le \theta \le \beta$.



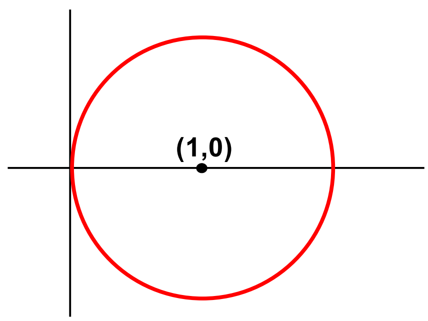
For example, the polar curve $r = 2\cos(\theta)$ can be seen to be a circle by plotting several points for a grid of θ values in the interval $[0,\pi]$ or by converting the equation to rectangular form via the relations between the two coordinate systems:

$$r = 2\cos(\theta)$$

$$\Rightarrow \sqrt{x^2+y^2} = 2 \times \frac{x}{\sqrt{x^2+y^2}}$$

$$\Rightarrow x^2+y^2 = 2x$$

$$\Rightarrow (x-1)^2 + y^2 = 1$$



As the angle θ goes from zero to $\pi/2$ radians, the point whose polar coordinates are r, θ traces out the top semi-circle in the figure from $(2,0)$ to $(0,0)$.

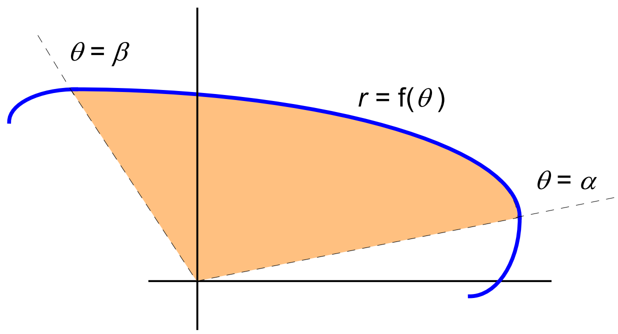
As the angle θ goes from $\pi/2$ to π radians, r is negative and the point whose polar coordinates are r, θ traces out the bottom semi-circle in the figure from $(0,0)$ to $(2,0)$.

The result is a circle of radius 1 centered on the point $(1,0)$.

Theory

Area enclosed by a polar curve

Consider the shaded region in the figure below, which is trapped between the rays $\theta = \alpha$, $\theta = \beta$ and the polar curve $r = f(\theta)$.



The area of this region can be computed as:

$$\int_{\theta=\alpha}^{\beta} \frac{f^2(\theta)}{2} d\theta$$

This formula arises from the fact that the area of a sector of a circle of radius r and angle $d\theta$ is given by:

$$\text{Area} = \underbrace{\left(\pi r^2\right)}_{\substack{\text{area of} \\ \text{full circle}}} \times \underbrace{\left(\frac{d\theta}{2\pi}\right)}_{\substack{\text{fraction} \\ \text{of the circle}}} = \frac{r^2}{2} \times d\theta$$

For example , the polar curve $r = f(\theta) = 5$ is a circle of radius 5 centered at the origin. (To see this, replace r by $\sqrt{x^2 + y^2}$ and then square both sides.)

As θ goes from 0 to 2π radians, the point whose polar coordinates are r, θ traces out the perimeter of this circle once. The area of this circle should be $\pi r^2 = 25\pi$. Let's see if this agrees with the area formula above:

$$\text{Area} = \int_{\theta=\alpha}^{\beta} \frac{f^2(\theta)}{2} d\theta = \int_0^{2\pi} \frac{5^2}{2}\, d\theta = \frac{25}{2} \times 2\pi = 25\pi$$

Theory

Length of a polar curve

How can we calculate the length of the polar curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$?

The length can be computed from our formula for distance along a parametric curve if we view θ as the parameter instead of the usual t. The parametric equations are given by:

$$x = r\cos(\theta) = f(\theta)\cos(\theta)$$

$$y = r\sin(\theta) = f(\theta)\sin(\theta)$$

The distance traveled along this curve as θ goes from α to β is calculated as follows with the parametric curve formulas:

$$\begin{aligned}
\text{Distance } &= \int_{\theta=\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \int_{\theta=\alpha}^{\beta} \sqrt{\left(\frac{d\,f(\theta)\cos(\theta)}{d\theta}\right)^2 + \left(\frac{d\,f(\theta)\sin(\theta)}{d\theta}\right)^2}\, d\theta \\
&= \int_{\theta=\alpha}^{\beta} \sqrt{\left(f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)\right)^2 + \left(f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)\right)^2}\, d\theta \\
&= \int_{\theta=\alpha}^{\beta} \sqrt{\left(f'(\theta)\right)^2 + \left(f(\theta)\right)^2}\, d\theta
\end{aligned}$$

Let's try this out on the circular polar curve and see if we can produce an answer that is consistent with the circumference of a circle.

For example, let $r = f(\theta) = 5$ where $0 \le \theta \le 2\pi$.

Using the distance formula above, we have:

$$\begin{aligned}
\text{distance } &= \int_{\theta=\alpha}^{\beta} \sqrt{\left(f'(\theta)\right)^2 + \left(f(\theta)\right)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{0^2 + 5^2}\, d\theta \\
&= 5\int_0^{2\pi} d\theta = 10\pi
\end{aligned}$$

And this reproduces the formula for the circumference of a circle:

$$\text{Circumference} = 2\pi r = 10\pi$$

Worked examples

Example 6.1

Reference: May 2000, Question 12

Let C be the curve defined by $x=\sin t+t$ and $y=\cos t-t$, $t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Find an equation of the tangent line to C at $(0,1)$.

(A) $y=1$

(B) $y=1+2x$

(C) $y=1-2x$

(D) $y=1-x$

(E) $y=1-\frac{1}{2}x$

Solution

The first step is to determine the time that the location is $(0,1)$:

$$\begin{matrix} 0=\sin t+t \\ 1=\cos t-t \end{matrix} \quad \Rightarrow \quad t=0$$

Now we need the tangent slope at this same point:

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-\sin t-1}{\cos t+1}\underbrace{=}_{\substack{\text{plug in}\\ \text{t=0}}}\frac{-0-1}{1+1}=-\frac{1}{2}$$

From the point-slope formula for a line, we see that an equation of the tangent line at $(0,1)$ is:

$$y-1=-\frac{1}{2}(x-0) \quad \Rightarrow \quad y=1-\frac{1}{2}x$$

So, the correct answer is **E**.

Worked examples

Example 6.2

Reference: BPP

For the curve in Example 6.1, calculate the speed at the instant the motion is located at $(0,1)$.

(A) 1/2
(B) $\sqrt{2}$
(C) 2
(D) $\sqrt{5}$
(E) 5

Solution

The location is $(0,1)$ at time $t=0$.

The speed is the length of the velocity vector:

$$\mathbf{v}(t) = \langle \cos t + 1, -\sin t - 1 \rangle$$

$$\Rightarrow \mathbf{v}(0) = \langle 1+1, -0-1 \rangle = \langle 2, -1 \rangle$$

$$\Rightarrow s'(0) = |\mathbf{v}(0)| = |\langle 2, -1 \rangle| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

Worked examples

Example 6.3 *Reference: May 2000, Question 21*

A ball rolls along the polar curve defined by $r = \sin(\theta)$.

The ball starts at $\theta = 0$ and ends at $\theta = \frac{3\pi}{4}$.

Calculate the distance that the ball travels.

(A) $\frac{3\pi}{8}$

(B) $\frac{3\pi}{4}$

(C) $\frac{9\pi}{8}$

(D) $\frac{3\pi}{2}$

(E) $\frac{15\pi}{8}$

Solution

We derived a formula earlier for the distance traveled along a polar curve $r = f(\theta)$:

$$\text{Distance} = \int_{\theta=\alpha}^{\beta} \sqrt{\left(f'(\theta)\right)^2 + \left(f(\theta)\right)^2}\, d\theta$$

$$= \int_0^{0.75\pi} \sqrt{\left(\sin'(\theta)\right)^2 + \left(\sin(\theta)\right)^2}\, d\theta$$

$$= \int_0^{0.75\pi} \sqrt{\left(\cos(\theta)\right)^2 + \left(\sin(\theta)\right)^2}\, d\theta$$

$$= \int_0^{0.75\pi} \sqrt{1}\, d\theta = 0.75\pi$$

So, the correct answer is **B**.

Worked examples

Example 6.4

Reference: BPP

Calculate the area enclosed by the polar curve $r = \sin(\theta)$ and the rays $\theta = 0$ and $\theta = 0.75\pi$.

(A) $\dfrac{3\pi}{8}$

(B) $\dfrac{2\pi+3}{8}$

(C) $\dfrac{3\pi+2}{16}$

(D) $\dfrac{3\pi}{4}$

(E) $\dfrac{3\pi+2}{4}$

Solution

We derived a formula earlier for the area enclosed by the polar curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$:

$$\text{Area} = \int_{\alpha}^{\beta} \frac{f(\theta)^2}{2} d\theta = \int_{0}^{0.75\pi} \frac{\sin^2(\theta)}{2} d\theta$$

Using integration by parts, with $u = \sin(\theta)$ and $dv = \sin(\theta)d\theta$, we have:

$$du = \cos(\theta)d\theta, \quad v = -\cos(\theta)$$

$$\int \sin^2(\theta)d\theta = -\cos(\theta)\sin(\theta) + \int \underbrace{\cos^2(\theta)}_{1-\sin^2(\theta)} d\theta$$

$$\Rightarrow 2\int \sin^2(\theta)d\theta = -\cos(\theta)\sin(\theta) + \int 1\, d\theta$$

$$\Rightarrow \int \sin^2(\theta)d\theta = \frac{-\cos(\theta)\sin(\theta)+\theta}{2}$$

$$\Rightarrow \int_0^{0.75\pi} \sin^2(\theta)d\theta = \frac{-\left(-\sqrt{2}/2\right)\left(\sqrt{2}/2\right)+3\pi/4}{2} = \frac{2+3\pi}{16}$$

So, the correct answer is **C**.

Note: This problem can also be solved using the trigonometric identity:

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \Rightarrow \int_0^{0.75\pi} \sin^2(\theta)d\theta = \left.\frac{\theta-0.5\sin(2\theta)}{2}\right|_0^{0.75\pi} = \frac{3\pi}{8} - \frac{0.5\times -1}{2} = \frac{3\pi+2}{8}$$

Practice questions

Question 6.1

Reference: November 2000, Question 15

Let *C* be the curve in $\mathbb{R}^3$ defined by $x = t^2, y = 4t^{3/2}, z = 9t$ for $t \geq 0$.

Calculate the distance along the curve from $(1,4,9)$ to $(16,32,36)$.

(A) 6
(B) 33
(C) 42
(D) 52
(E) 597

Question 6.2

Reference: November 2000, Question 33

Let *C* be the curve defined by the polar function $r = 2 + \cos(\theta)$. The vertices of triangle *PQR* are the points on *C* corresponding to $\theta = 0$, $\theta = \pi$ and $\theta = \pi/3$.

Calculate the area of *PQR.*

(A) 2
(B) $\dfrac{5\sqrt{3}}{4}$
(C) $\dfrac{5}{2}$
(D) 4
(E) $\dfrac{5\sqrt{3}}{2}$

Question 6.3

Reference: May 2001, Question 3

The coordinates of an object moving in $\mathbb{R}^2$ are:

$$x = 4\sin\frac{t}{2} \qquad y = 2t\cos t \qquad \text{for time } t > 0$$

Calculate the length of the velocity vector of the object at time $t = \pi/2$.

(A) $\sqrt{2}$
(B) π
(C) $\sqrt{\pi^2 + 2}$
(D) $\sqrt{\pi^2 + 4}$
(E) $\pi + \sqrt{2}$

Practice questions

Question 6.4

Reference: May 2001, Question 15

Let C be the curve defined by:

$$x = 2t^2 + t - 1,\ y = t^2 - 3t + 1 \quad \text{for } -\infty < t < \infty.$$

What is the slope of the line tangent to C at $(0,5)$?

(A) -5
(B) -1
(C) 3/5
(D) 5/3
(E) 7

Question 6.5

Reference: November 2001, Question 6

Let C be the curve defined by the parametric equations $x = t^2 + t$, $y = t^2 - 1$ for $-\infty < t < \infty$.

Determine the time at which the line tangent to the graph of C is parallel to the line $5y - 4x = 3$.

(A) $-1/10$
(B) 2/5
(C) 5/8
(D) 5/3
(E) 2

Question 6.6

Reference: November 2001, Question 23

Let R be the region bounded by the polar curve $r = \sin(\theta) + \sqrt{3}\cos(\theta)$.

Which of the following represents the area of the subset of r to the left of the line $\theta = \frac{\pi}{2}$?

(A) $\frac{1}{2}\int_0^{\pi/2} r^2\, d\theta$

(B) $\frac{1}{2}\int_{\pi/2}^{\pi} r^2\, d\theta$

(C) $\frac{1}{2}\int_{\pi/2}^{2\pi/3} r^2\, d\theta$

(D) $\frac{1}{2}\int_{\pi/2}^{2\pi} r^2\, d\theta$

(E) $\frac{1}{2}\int_{\pi/3}^{\pi/2} r^2\, d\theta$

