Course 1 Key Concepts

Calculus

Lesson 7

Multivariable Differential Calculus

Overview

In this lesson, we will discuss the differential calculus of a function $z = f(x, y)$ of two variables. Many of the concepts that you will see in this setting are extensions of the concepts that we studied in single-variable calculus, for a function $y = f(x)$.

We'll study topics such as limits of a two-variable function, continuity, partial derivatives, and optimization problems. Although the material may feel rather abstract, it does have practical applications. For example, suppose that a company will spend \$$x$ on the development of a new insurance product and \$$y$ on the promotion of this product. From past experience the company might be able to project the resulting sales volume of the new product in the first 5 years after its launch as $z = f(x, y)$. Optimization techniques could be used to find the optimal allocation of finite resources between development and promotion.

BPP Learning Objectives

This lesson covers the following BPP learning objectives:

(C22) *Understand the concepts of limits, continuity, partial derivatives, gradient vectors, directional derivatives, and tangent planes in multivariable calculus.*

(C23) *Apply the concepts in (C22) to solve problems set in a business context, including max-min problems.*

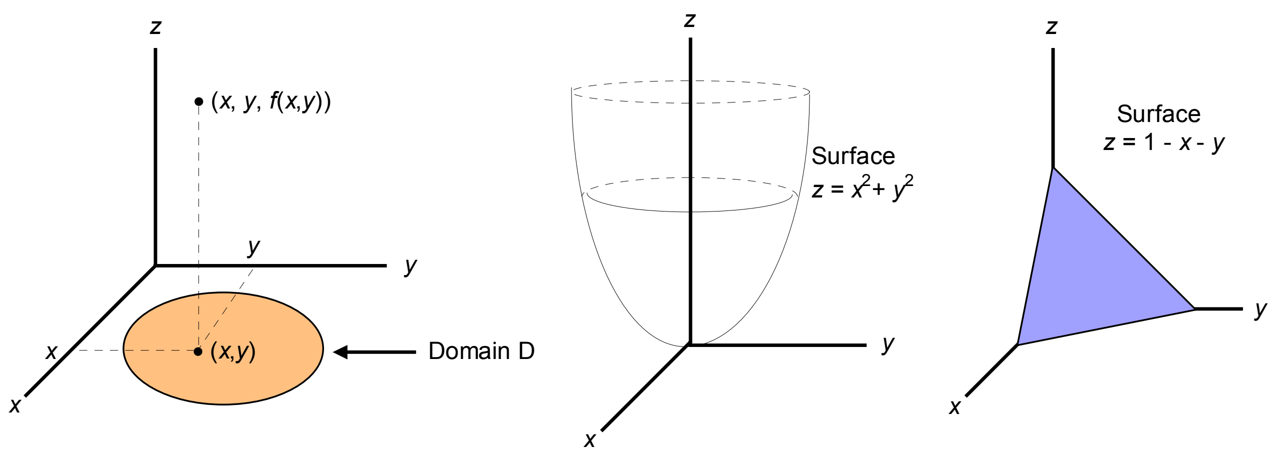
Theory

Functions of two variables and their graphs

A **function of two variables** assigns to each point (x, y) in its domain D (a set in $\mathbb{R}^2$) a unique real number z that is denoted by $f(x, y)$. This indicates that the assigned value depends on (and can be calculated from) the **independent variables** x and y.

In some applications a domain will be explicitly stated for the function. On the other hand, if a function such as $z = (x^2 - y^2)/(x+1)$ is given without mentioning the domain, then it is typically assumed that the domain is the set of all points in the plane for which the formula makes sense (here it is all points except those lying on the line $x = -1$).

The graph of a two-variable function $z = f(x, y)$ is a surface in 3-space consisting of all points $(x, y, f(x, y))$ where (x, y) varies over the domain *D*.



The simplest surface in 3-space is a **plane**. A plane consists of all points (x, y, z) satisfying a linear equation:

$$ax + by + cz = d$$

If the coefficient *c* is non-zero, then the equation can be rewritten in the functional form:

$$z = \frac{d - ax - by}{c}$$

More complicated graphs $z = f(x, y)$ can be constructed by sketching the intersection of the graph with various types of planes. For example, the graph of the function $z = x^2 + y^2$ intersects the plane $z = c$ (a plane parallel to the *xy*-plane) in a circle $c = x^2 + y^2$ if $c > 0$ (see the middle figure above). Planes of the type $X = x_0$ and $Y = y_0$ are perpendicular to the *xy*-plane. They will be used in the geometric interpretation of partial derivatives.

Theory

Limits and continuity

The phrase "all (x,y) near (x_0,y_0)" refers to the set consisting of all points (x,y) satisfying the inequality:

$$(x-x_0)^2+(y-y_0)^2<\delta \qquad \text{for a "small" positive number } \delta$$

These points are inside a circle of radius δ that is centered at the point (x_0,y_0).

Definitions

The **limit** of a function $z=f(x,y)$ as (x,y) approaches (x_0,y_0) is denoted $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$.

The function $z=f(x,y)$ is said to have a **limit** L as (x,y) approaches (x_0,y_0) if the function value $f(x,y)$ gets closer to L as the point (x,y) nears (x_0,y_0) without touching it.

Note: To consider this limit, all points near (x_0,y_0), except possibly (x_0,y_0) itself, must be in the domain.

The function $z=f(x,y)$ is said to be **continuous** at the point (x_0,y_0) if the function is defined at this point, the limit of the function at this point exists, and if this limit is equal to the function value, *ie*:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0).$$

Just as in single-variable calculus, the concepts of limit and continuity are intertwined.

In two-variable calculus, limit rules and rules for the combination of continuous functions are analagous to their single-variable counterparts. Also, any algebraic function of two variables (one whose formula is made up from x, y, constants and the operations of addition, subtraction, division, multiplication and the taking of *n*-th roots) is continuous wherever it is defined. However, in two-variable calculus some limit problems can be very subtle.

For example, the algebraic function $z=\left(x^2-y^2\right)\big/\left(x^2+y^2\right)$ is defined everywhere except at the origin $(0,0)$. Hence it is continuous except at the origin. So if $(x_0,\ y_0)\neq(0,0)$, we have:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)=\frac{x_0^2-y_0^2}{x_0^2+y_0^2}$$

Now consider the limit of this same function at the origin, *ie* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

If we attempt to apply the quotient rule for limits it results in the undefined expression 0/0.

In single-variable calculus we can deal with problems of this type by factoring and canceling or by applying L'Hopital's Rule.

For two-variable limit problems of the "0/0 type" these single-variable tricks are not usually available.

Theory

However, sometimes we can show that a limit fails to exist by approaching the origin along linear paths such as $y = mx$ and obtaining different single-variable limits for different paths.

If $y = mx$, then for non-zero x we have the following:

$$f(x,y) = f(x,mx) = \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1-m^2}{1+m^2}$$

$$\Rightarrow \lim_{\substack{x\to 0\\ y=mx}} f(x,y) = \frac{1-m^2}{1+m^2}$$

Since the limit along the path $y = mx$ differs as m differs, the two-variable limit of $f(x,y)$ as (x,y) approaches the origin cannot exist.

Partial derivatives

In general, a **partial derivative** of a function of several independent variables is defined as the instantaneous rate of change of the function with respect to one particular independent variable while all of the other independent variables are held fixed.

This idea makes partial derivative calculation virtually identical to ordinary derivative calculation in single-variable calculus. Partial derivatives have a limit-definition that is the basis of all theory and the foundation of their geometric interpretation.

We will explore these ideas for a function of two variables.

Definition

Suppose that the function $z = f(x,y)$ is defined near the point (x_0, y_0). The **partial derivatives** of this function with respect to *x* and *y* are defined by the following limits (if they exist):

$$\underbrace{\frac{\delta f}{\delta x}(x_0,y_0) = f_x(x_0,y_0)}_{\text{standard notation}} = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$$

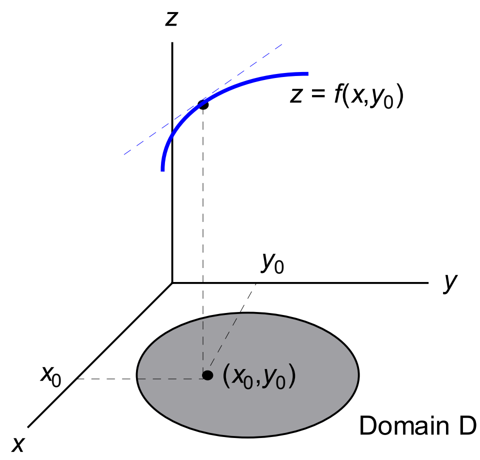
$$\underbrace{\frac{\delta f}{\delta y}(x_0,y_0) = f_y(x_0,y_0)}_{\text{standard notation}} = \lim_{\Delta y\to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0,y_0)}{\Delta y}$$

Theory

The figure on the right pictures the intersection of the plane $Y = y_0$ with the surface $z = f(x,y)$ (*ie* the graph of this function).

This intersection is the curve $z = f(x, y_0)$ in the figure.

The partial derivative of the function $z = f(x,y)$ with respect to x at the point (x_0, y_0), $f_x(x_0, y_0)$, is the slope of the dotted tangent line to the cross-sectional curve.



So, let's summarize the geometric meaning of partial derivatives:

- $f_x(x_0, y_0)$ is the tangent slope at the point $(x_0, y_0, f(x_0, y_0))$ on the curve $z = f(x, y_0)$ formed by the intersection of the plane $Y = y_0$ with the surface $z = f(x,y)$.
- $f_y(x_0, y_0)$ is the tangent slope at the point $(x_0, y_0, f(x_0, y_0))$ on the curve $z = f(x_0, y)$ formed by the intersection of the plane $X = x_0$ with the surface $z = f(x,y)$.

A partial derivative is just like an ordinary derivative with respect to one independent variable, while the other independent variables are held fixed. For example, using the derivative rules from single-variable calculus:

$$z = x^2 + 2xy \Rightarrow \frac{\delta z}{\delta x} = 2x + 2y$$

$$z = \frac{x^2 + 2xy}{x - 2y} \Rightarrow \frac{\delta z}{\delta x} = \frac{(x-2y)\cdot\dfrac{\delta\left(x^2+2xy\right)}{\delta x} - \left(x^2+2xy\right)\cdot\dfrac{\delta(x-2y)}{\delta x}}{(x-2y)^2}$$

$$= \frac{(x-2y)(2x+2y) - \left(x^2+2xy\right)}{(x-2y)^2}$$

$$z = \frac{x^2 + 2xy}{x - 2y} \Rightarrow \frac{\delta z}{\delta y} = \frac{(x-2y)\cdot\dfrac{\delta\left(x^2+2xy\right)}{\delta y} - \left(x^2+2xy\right)\cdot\dfrac{\delta(x-2y)}{\delta y}}{(x-2y)^2}$$

$$= \frac{(x-2y)(2x) - \left(x^2+2xy\right)(-2)}{(x-2y)^2}$$

$$z = e^{-\left(x^2+y^2\right)} \Rightarrow \frac{\delta z}{\delta x} = e^{-\left(x^2+y^2\right)} \cdot \frac{\delta\left(-\left(x^2+y^2\right)\right)}{\delta x} = -2xe^{-\left(x^2+y^2\right)}$$

Theory

The concept of **differentiability** in calculus is supposed to reflect "smoothness."

In single-variable calculus, if the derivative of a function exists at every point in an interval, then the graph of this function is smooth. In two-variable calculus, the existence of both partial derivatives is not enough to guarantee smoothness. However, if both of the partial derivatives of a function $z = f(x,y)$ exist and are continuous, then the graph is smooth and the function is said to be differentiable.

The gradient and directional derivatives of a two-variable function

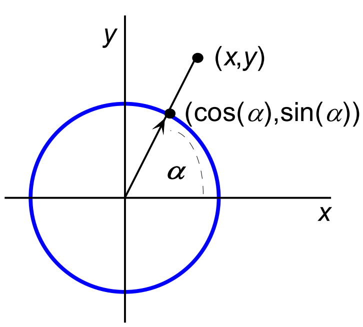
Definition

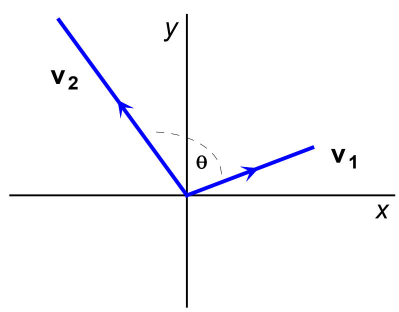
A **unit vector** is a vector with length 1.

- $\mathbf{u} = \langle x, y \rangle$ is a unit vector if $x^2 + y^2 = 1$
- If $\mathbf{v} = \langle x, y \rangle$ is any vector, then $\mathbf{u} = \dfrac{1}{|\mathbf{v}|}\mathbf{v} = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$ is a unit vector parallel to $\mathbf{v}$.
- The vector $\mathbf{u} = \langle \cos(\alpha), \sin(\alpha) \rangle$ is a unit vector that is α radians in a counterclockwise direction from the positive *x*-axis (see the figure below).

The **dot product** of two vectors is defined by $\langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle = x_1 x_2 + y_1 y_2$

- $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$
- $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1||\mathbf{v}_2|\cos(\theta)$ where θ is the angle between the two vectors when their tails are plotted at the origin (see the figure below.)
- Two vectors are perpendicular if their dot product is zero (since $\cos(\pi/2) = 0$).





Theory

Let (x_0, y_0) be a point in the domain of a differentiable function $z = f(x,y)$. If we depart this point in a direction parallel to the unit vector **u**, then the instantaneous rate of change of the function with respect to distance is called the **directional derivative** of $z = f(x,y)$ at this point in the direction of **u**.

- If $\mathbf{u} = \langle 1,0 \rangle$ is the unit vector in the positive *x*-direction, then the directional derivative is the same as the partial derivative of the function with respect to *x*.
- If $\mathbf{u} = \langle 0,1 \rangle$ is the unit vector in the positive *y*-direction, then the directional derivative is the same as the partial derivative of the function with respect to *y*.

There is a limit definition of a directional derivative similar to the one for partial derivatives. However, a directional derivative is computed most easily in terms of a dot product with the gradient vector.

Definition

The **gradient** vector of a differentiable function $f(x,y)$ is denoted $\nabla f(x,y)$ and is a vector whose components are the partial derivatives of this function, *ie*:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

The tail of the gradient is at the point (x_0, y_0).

The **directional derivative** of $f(x,y)$ at (x_0, y_0) in the direction **u** is denoted by $D_{\mathbf{u}} f(x_0, y_0)$ and is the rate of change of the function with respect to distance when we depart the point (x_0, y_0) in the direction of **u**.

The vector **u** is assumed to be a unit vector.

Properties of directional derivatives

(i) $D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = |\nabla f(x_0, y_0)| \cos(\theta)$ where θ is the angle between the gradient vector and the direction **u**.

(ii) At the point (x_0, y_0), the direction in which the directional derivative is greatest is the direction of the gradient vector itself (*ie* $\cos(\theta = 0) = 1$). The directional derivative in this direction equals $|\nabla f(x_0, y_0)|$.

(iii) The level curve of the function $f(x,y)$ through the point (x_0, y_0) consists of all (x,y) for which $f(x,y) = f(x_0, y_0)$. The gradient at (x_0, y_0) is perpendicular to the level curve through this point.

Theory

For example, let's analyze the directional derivatives of the function $f(x,y)=2x^2+y^2$ at the point $(1,1)$ in its domain. The gradient vector at a general point is:

$$\nabla f(x,y)=\langle 4x,2y\rangle$$

So we have $\nabla f(1,1)=\langle 4,2\rangle$.

A unit vector parallel to the direction of the gradient is:

$$\mathbf{u}=\left\langle \frac{4}{\sqrt{20}},\frac{2}{\sqrt{20}}\right\rangle.$$

If $\mathbf{u}=\langle a,b\rangle$ is a unit vector, then the vectors $\langle -b,a\rangle$ and $\langle b,-a\rangle$ are unit vectors perpendicular to **u** since they have a dot product of zero with **u**.

$$\mathbf{u}=\left\langle \frac{4}{\sqrt{20}},\frac{2}{\sqrt{20}}\right\rangle \Rightarrow D_{\mathbf{u}}f(1,1)=\langle 4,2\rangle\bullet\mathbf{u}=\sqrt{18}$$
$$\mathbf{u}=\left\langle \frac{-2}{\sqrt{20}},\frac{4}{\sqrt{20}}\right\rangle \Rightarrow D_{\mathbf{u}}f(1,1)=\langle 4,2\rangle\bullet\mathbf{u}=0$$
$$\mathbf{u}=\left\langle \frac{2}{\sqrt{20}},\frac{-4}{\sqrt{20}}\right\rangle \Rightarrow D_{\mathbf{u}}f(1,1)=\langle 4,2\rangle\bullet\mathbf{u}=0$$
$$\mathbf{u}=\left\langle \frac{-4}{\sqrt{20}},\frac{-2}{\sqrt{20}}\right\rangle \Rightarrow D_{\mathbf{u}}f(1,1)=\langle 4,2\rangle\bullet\mathbf{u}=-\sqrt{18}$$

The level curve through the point $(1,1)$ is the set of all points (x,y) satisfying $2x^2+y^2=f(x,y)=f(1,1)=3$.

At the point (x,y) on this level curve, the tangent slope is $\frac{dy}{dx}=\frac{-4x}{2y}$ (implicit differentiation).

So the tangent slope at $(1,1)$ is $-4/2=-2$.

The "slope" (*ie* rise/run) of the gradient vector $\langle 4,2\rangle$ at this point is $2/4=0.5$. Since the product of these two slopes is -1, we can see that $\nabla f(1,1)$ is perpendicular to the tangent line at $(1,1)$ on the level curve.

Theory

The chain rule in multivariable calculus

The chain rule is concerned with how derivatives or partial derivatives are computed for composite functions.

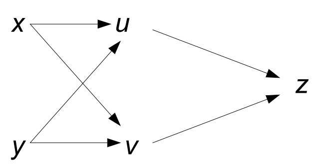
In single-variable calculus, if *y* depends on *x* and *z* depends on *y*, then *z* depends on *x* through the composite function and the chain rule can be written as:

$$\frac{dz}{dx} = \frac{dy}{dx} \cdot \frac{dz}{dy}$$

To write down the most general form of the chain rule in multivariable calculus requires quite a bit of notation and the formula obscures how simple the process is. So, we'll present the theory in an informal and conceptually simple way.

Let's consider the following example in order to present the technique.

The intermediate variables *u* and *v* are assumed to depend on the initial variables *x* and *y*. The final variable *z* is assumed to depend on the intermediate variables *u* and *v*. Via the composition of these functions, it follows that *z* depends on *x* and *y*.



How can we compute the partial derivatives of *z* with respect to *x* and *y*?

Keep in mind that differentiable functions are approximately linear over a small portion of the domain. So let's see how the chain rule would work for the calculation of $\delta z / \delta x$ if all of the functions in the above diagram were linear.

$$u = a_1 x + b_1 y + c_1 \quad , \quad v = a_2 x + b_2 y + c_2$$

$$z = du + ev + f \Rightarrow z = d\left(a_1 x + b_1 y + c_1\right) + e\left(a_2 x + b_2 y + c_2\right) + f = \left(a_1 d + a_2 e\right)x + \left(b_1 d + b_2 e\right)y + g$$

$$\Rightarrow \frac{\delta z}{\delta x} = a_1 d + a_2 e = \frac{\delta u}{\delta x} \cdot \frac{\delta z}{\delta u} + \frac{\delta v}{\delta x} \cdot \frac{\delta z}{\delta v}$$

This formula derived in the linear case turns out to be the general formula. A verbal description of this formula in terms of rates of change along paths from the initial variable through the intermediate variables to the final variables will suffice in any case:

Chain Rule for Multivariable Calculus – Summary

(i) Consider all paths from the initial variable through an intermediate variable to the final variable.

(ii) For each such path multiply the rates of change along both segments of the path.

(iii) Sum these products over all possible paths.

Theory

The differential and the tangent plane in two-variable calculus

Both of these ideas are concerned with linear approximation of a differentiable function $z = f(x,y)$ near a point (x_0, y_0) in the domain. The equation of the **tangent plane** at the point $(x_0, y_0, f(x_0, y_0))$ on the graph of $z = f(x,y)$ is given by:

$$z_{\tan} = f(x_0, y_0) + f_x(x_0, y_0)\cdot(x - x_0) + f_y(x_0, y_0)\cdot(y - y_0)$$

The properties of the tangent plane are as follows:

- It passes through the point $(x_0, y_0, f(x_0, y_0))$
- It has the same partial derivatives at (x_0, y_0) as $z = f(x,y)$

The tangent plane is a two-variable extension of the concept of the tangent line in single-variable calculus.

For (x,y) near (x_0, y_0), the tangent plane can be used as a linear approximation to the function $y = f(x,y)$:

$$f(x,y) \approx z_{\tan} = f(x_0, y_0) + f_x(x_0, y_0)\cdot(x - x_0) + f_y(x_0, y_0)\cdot(y - y_0)$$

When this approximation is rewritten as follows you will be able to see the differential dz as an approximation to Δz, the change in z as you pass from (x_0, y_0) to a nearby point (x,y):

$$\underbrace{f(x,y) - f(x_0, y_0)}_{\Delta z} \approx \underbrace{f_x(x_0, y_0)\cdot\underbrace{(x - x_0)}_{\Delta x} + f_y(x_0, y_0)\cdot\underbrace{(y - y_0)}_{\Delta y}}_{\text{defined to be the differential } dz}$$

$$\Rightarrow \qquad \Delta z \approx dz = f_x(x_0, y_0)\cdot\Delta x + f_y(x_0, y_0)\cdot\Delta y$$

Let's see this theory in a simple example.

Let $f(x,y) = xy$.

Then we have the following:

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(xy) = (x + \Delta x)(y + \Delta y) - xy = \underbrace{y\,\Delta x + x\,\Delta y}_{dz} + \underbrace{(\Delta x)(\Delta y)}_{\substack{\text{error in the} \\ \text{approximation} \\ \Delta z \approx dz}}$$

Theory

Constrained max-min problems for two-variable functions

In some Course 1 exam questions, the problem is to find the extreme values of the function $z = f(x,y)$ when the points (x,y) are restricted to lie on a curve in the domain. The curve might be given in the form $y = g(x)$, $a \le x \le b$ or in the parametric form $x = g(t), y = h(t)$, $t_0 \le t \le t_1$. In either case, the equation(s) describing the curve is/are substituted into $z = f(x,y)$, and the result is a single-variable function. We can then apply the standard max-min theory from single-variable calculus.

For example, let's find the minimum and maximum values of the function $f(x,y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 25$. So, the point (x,y) lies on a circle of radius 5 centered at the origin.

Method 1 – direct substitution

We have:

$$f(x,y) = x^2 + 2y^2 = 25 + y^2 \quad \text{where } -5 \le y \le 5$$

$$h(y) = 25 + y^2 \Rightarrow h'(y) = 2y \Rightarrow h'(0) = 0 \Leftrightarrow y = 0$$

Valuing the function at the critical point $y = 0$ and the endpoints $y = -5$ and $y = 5$, we find:

$h(-5) = h(5) = 50$ *ie* maximum value at endpoints

$h(0) = 25$ *ie* minimum value at critical point

Method 2 – parametric form

We can parameterize the circle constraint as follows:

$$x = 5\cos(t) \quad y = 5\sin(t) \quad 0 \le t \le 2\pi$$

$$h(t) = f(5\cos(t), 5\sin(t)) = 25\cos^2(t) + 50\sin^2(t) = 25 + 25\sin^2(t)$$

$$h'(t) = 50\sin(t)\cos(t) \Rightarrow h'(t) = 0 \Leftrightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

Comparing the values of $h(t)$ at the endpoints $t = 0, 2\pi$ and the interior critical points $t = \pi/2, \pi, 3\pi/2$, we can see again that the minimum value is 25 (at $t = 0, \pi, 2\pi$) and the maximum is 50 (at $t = \pi/2$, $3\pi/2$).

These constrained max-min problems typically occur as one step in the process of finding the extreme values of the function $z = f(x,y)$ on a closed and bounded set. There is an alternate technique for solving these constrained problems that relies on Lagrange multipliers. The idea of this method is that at a critical point on the constraining curve the gradient vector of $z = f(x,y)$ must be perpendicular to the tangent direction on the curve.

Theory

Local max-min problems for two-variable functions

A **local maximum** or **local minimum** for the function $z=f(x,y)$ occurs at the point (x_0,y_0) if the point $(x_0,y_0,f(x_0,y_0))$ on the graph of $z=f(x,y)$ lies at the top of a "smooth mountain" or at the bottom of a "smooth valley."

The tangent plane at such a point must be horizontal (*ie* $f_x(x_0,y_0)=0$, $f_y(x_0,y_0)=0$). In two-variable calculus it is a bit trickier to determine if a local max or local min occurs at a **critical point** (one where both partial derivatives are zero).

The following criterion is analogous to the second derivative criterion in single-variable calculus. For a twice-differentiable function $z=f(x,y)$ the second order partial derivatives are defined as partial derivatives of the first order partial derivatives:

$$f_{xx}(x,y)=\frac{\delta f_x}{\delta x} \quad , \quad f_{yy}(x,y)=\frac{\delta f_y}{\delta y} \quad , \quad f_{xy}=\frac{\delta f_x}{\delta y}=f_{yx}=\frac{\delta f_y}{\delta x}$$

Criterion for local extreme points

Suppose that $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$.

The **discriminant** at this critical point is defined as:

$$\Delta=f_{xx}(x_0,y_0)\cdot f_{yy}(x_0,y_0)-\left(f_{xy}(x_0,y_0)\right)^2$$

(i) If $\Delta>0$ and $f_{xx}(x_0,y_0)>0$, then a local minimum occurs at the critical point.

(ii) If $\Delta>0$ and $f_{xx}(x_0,y_0)<0$, then a local maximum occurs at the critical point.

(iii) If $\Delta<0$, then a saddle point ("mountain pass") occurs at the critical point.

(iv) If $\Delta=0$, then no conclusion is possible – the critical point could be a local extreme point or a saddle point.

For example, let's find the local extreme points of the function

$$z=x^2+2y^2+2xy-2x+5.$$

Critical points occur where both partial derivatives are zero:

$$0=f_x=2x+2y-2 \quad \text{and} \quad 0=f_y=4y+2x \quad \Rightarrow (x,y)=(2,-1)$$

$$f_{xx}=2,\quad f_{yy}=4,\quad f_{xy}=2 \Rightarrow \Delta(x,y)=f_{xx}\cdot f_{yy}-\left(f_{xy}\right)^2=8-4>0$$

From criterion (i) above, we see that a local minimum of this function occurs at the lone critical point $(2,-1)$.

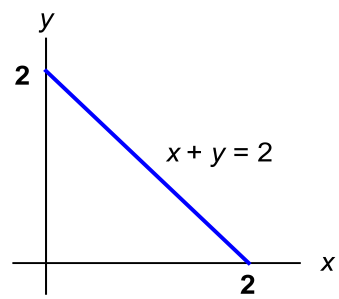
Theory

Absolute max-min problems for two-variable functions

A set D in $\mathbb{R}^2$ is **closed** if all of the edge (boundary) points belong to the set, and **bounded** if the set is finite in width. The point (x_0, y_0) is an **interior point** if it lies inside the boundary curve.

> **Theorem**
>
> Let $z = f(x, y)$ be a continuous function.
>
> Then z has both an absolute maximum and an absolute minimum on a closed and bounded set *D* in $\mathbb{R}^2$.
>
> The absolute extreme values occur at either a critical point in the interior of *D* or on the boundary (edge) of *D*.

For example, let $z = xy(2x + 3y - 6)$ be defined on the closed and bounded triangular domain having vertices at the points $(0,0)$, $(2,0)$, $(0,2)$. See the figure on the right.



To find the absolute extreme values within this domain, the first step is to solve the simultaneous equations:

$$f_x = 0, \ f_y = 0$$

to find any critical points in the interior of this triangle.

$$0 = f_x = y(4x + 3y - 6), \qquad 0 = f_y = x(2x + 6y - 6)$$

$$\Rightarrow \ (x, y) = \left(1, \tfrac{2}{3}\right)$$

The function value at the interior critical point is $-4/3$. The function is identically zero on both the vertical and horizontal boundary lines of the triangle. On the boundary segment $x + y = 2$, $0 \le x \le 2$ we have:

$$y = 2 - x \Rightarrow h(x) = f(x, y) = f(x, 2 - x) = -x^2(2 - x) = -2x^2 + x^3$$

$$\Rightarrow h'(x) = -4x + 3x^2 = x(-4 + 3x) \Rightarrow h'(x) = 0 \Leftrightarrow x = 4/3$$

$$h(0) = h(2) = 0$$

$$h(4/3) = -32/27$$

Comparing the function value at the interior critical point to the extreme boundary values, we see that the minimum value is $-4/3$ (at the interior critical point) and the maximum value is zero (all along the horizontal and vertical segments of the boundary triangle).

Theory

Implicit differentiation

One type of surface in 3-space is the graph of a function $z = f(x,y)$. Another type of surface consists of all points (x,y,z) satisfying an equation in three variables. For example, the solutions of the equation $x^2 + y^2 + z^2 = 50$ lie on a spherical surface centered at the origin and having radius $\sqrt{50}$.

Let's study two approaches to calculate the tangent plane at the point $(3,4,-5)$ on this sphere.

Method 1 – Find *z* in terms of *x* and *y*

The equation of the sphere can be solved for *z* in terms of *x* and *y*. We have $z = \pm\sqrt{50 - x^2 - y^2}$.

The point $(3,4,-5)$ lies on the graph of the function $z = f(x,y) = -\sqrt{50 - x^2 - y^2}$.

To find the equation of the tangent plane at this point we need 2 partial derivatives:

$$f_x(x,y) = -\frac{1}{2}\left(50 - x^2 - y^2\right)^{-1/2} \cdot \frac{\delta\left(50 - x^2 - y^2\right)}{\delta x} = \frac{x}{\sqrt{50 - x^2 - y^2}} \quad \Rightarrow f_x(3,4) = \frac{3}{\sqrt{25}} = \frac{3}{5}$$

$$f_y(x,y) = -\frac{1}{2}\left(50 - x^2 - y^2\right)^{-1/2} \cdot \frac{\delta\left(50 - x^2 - y^2\right)}{\delta y} = \frac{y}{\sqrt{50 - x^2 - y^2}} \quad \Rightarrow f_y(3,4) = \frac{4}{\sqrt{25}} = \frac{4}{5}$$

So, the equation of the tangent plane is:

$$z_{\tan} = f(x_0, y_0) + f_x(x_0, y_0)\cdot(x - x_0) + f_y(x_0, y_0)\cdot(y - y_0) = -5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$$

Method 2 – Implicit differentiation

When computing partial derivatives implicitly, we simply assume that *z* depends on both *x* and *y* and then partially differentiate both sides of the equation:

$$x^2 + y^2 + z^2 = 50 \;\Rightarrow\; \frac{\delta\left(x^2 + y^2 + z^2\right)}{\delta x} = \frac{\delta 50}{\delta x} \;\Rightarrow\; 2x + 0 + 2z\frac{\delta z}{\delta x} = 0 \;\Rightarrow\; \frac{\delta z}{\delta x} = \frac{-x}{z}$$

$$x^2 + y^2 + z^2 = 50 \;\Rightarrow\; \frac{\delta\left(x^2 + y^2 + z^2\right)}{\delta y} = \frac{\delta 50}{\delta y} \;\Rightarrow\; 0 + 2y + 2z\frac{\delta z}{\delta y} = 0 \;\Rightarrow\; \frac{\delta z}{\delta y} = \frac{-y}{z}$$

The difference in this method is that the partial derivatives are now expressed in terms of *x*, *y*, and *z*.

If we plug in $(x,y,z) = (3,4,-5)$, we'll see that we get the same values of the two partial derivatives as those obtained in the first method.

Worked examples

Example 7.1

Reference: May 2000, Question 37

Let S be the surface described by $f(x,y) = \arctan\left(\frac{y}{x}\right)$.

Determine an equation of the plane tangent to S at the point $\left(1,1,\frac{\pi}{4}\right)$.

(A) $z = \frac{\pi}{4} - \frac{1}{2}(x-1) - \frac{1}{2}(y-1)$

(B) $z = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$

(C) $z = \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$

(D) $z = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{2}(y-1)$

(E) $z = \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$

Solution

Recall that $\arctan'(x) = \frac{1}{1+x^2}$.

At $(x_0, y_0, z_0) = (1, 1, \pi/4)$, the tangent plane equation is given by:

$$\begin{aligned} z_{\tan} &= f(x_0,y_0) + f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0) \\ &= \frac{\pi}{4} + f_x(1,1)(x-1) + f_y(1,1)(y-1) \end{aligned}$$

where:

$$f_x(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\delta(y/x)}{\delta x} = \frac{-y/x^2}{1+\left(\frac{y}{x}\right)^2} = \frac{-y}{x^2+y^2} \Rightarrow f_x(1,1) = -\frac{1}{2}$$

$$f_y(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\delta(y/x)}{\delta y} = \frac{1/x}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2} \Rightarrow f_y(1,1) = \frac{1}{2}$$

So, the correct answer is **B**.

Worked examples

Example 7.2

Reference: May 2001, Question 8

The number of items produced by a manufacturer is given by $p = 100\sqrt{xy}$, where x is the amount of capital and y is the amount of labor.

At a particular point in time:

(i) the manufacturer has 2 units of capital

(ii) capital is increasing at a rate of 1 unit per month

(iii) the manufacturer has 3 units of labor

(iv) labor is decreasing at a rate of .5 units per month

Determine the rate of change in the number of items produced at the given time.

(A) 41
(B) 61
(C) 82
(D) 102
(E) 245

Solution

Here is an exercise in the multivariable chain rule.

There are two paths through the intermediate variables from t to p :

$$\begin{tikzcd} & x \arrow[dr] & \\ t \arrow[ur] \arrow[dr] & & p \\ & y \arrow[ur] & \end{tikzcd}$$

So we have:

$$\frac{dp}{dx} = 100\sqrt{y} \cdot \frac{1}{2\sqrt{x}} = 50\sqrt{\frac{y}{x}}$$

$$\frac{dp}{dy} = 100\sqrt{x} \cdot \frac{1}{2\sqrt{y}} = 50\sqrt{\frac{x}{y}}$$

$$\frac{dp}{dt} = \frac{dx}{dt} \cdot \frac{\delta p}{\delta x} + \frac{dy}{dt} \cdot \frac{\delta p}{\delta y} = (1)\left(50\sqrt{\frac{3}{2}}\right) + (-0.5)\left(50\sqrt{\frac{2}{3}}\right) = 40.825$$

So, the correct answer is **A**.

Worked examples

Example 7.3

Reference: BPP

The temperature in degrees at a point (x,y) is:

$$T(x,y)=100e^{-0.01(x^2+y^2)}$$

A bug at the point $(2,1)$ begins moving away from this point along a straight path toward the point $(5,5)$.

Calculate the rate of change of temperature with respect to distance (x and y in feet) along this path at the instant the bug departs from $(2,1)$.

(A) –6.227 degrees/foot

(B) –3.805 degrees/foot

(C) –2.384 degrees/foot

(D) +1.394 degrees/foot

(E) +5.633 degrees/foot

Solution

We need to calculate the directional derivative $D_{\mathbf{u}}T(2,1)$ where **u** is a unit vector pointing from $(2,1)$ toward $(5,5)$.

The displacement vector from the first point to the second point is $\mathbf{v}=\langle 5-2,5-1\rangle=\langle 3,4\rangle$.

A unit vector parallel to **v** is given by $\mathbf{u}=\dfrac{\mathbf{v}}{|\mathbf{v}|}=\dfrac{\langle 3,4\rangle}{\sqrt{3^2+4^2}}=\langle 0.6,0.8\rangle$.

Hence:

$$\frac{\delta T}{\delta x}=100e^{-0.01(x^2+y^2)}\times(-0.01\times 2x)$$

$$\frac{\delta T}{\delta y}=100e^{-0.01(x^2+y^2)}\times(-0.01\times 2y)$$

$$\Rightarrow \nabla T(2,1)=\left\langle 100e^{-0.05}\times -0.04\text{ , }100e^{-0.05}\times -0.02\right\rangle=\langle -3.805\text{ , }-1.902\rangle$$

$$\Rightarrow D_{\mathbf{u}}T(2,1)=\nabla T(2,1)\bullet\mathbf{u}=\langle -3.805\text{ , }-1.902\rangle\bullet\langle 0.6,\ 0.8\rangle=-3.805\text{ degrees/foot}$$

So, the correct answer is **B**.

Practice questions

Question 7.1 *Reference: May 2001, Question 18*

The temperature of a particle located at the point (u,v) is $f(u,v)=e^{uv}$. The location is determined by two inputs *x* and *y* such that:

$$\frac{\delta u}{\delta x}=2y \quad \frac{\delta u}{\delta y}=2x \quad \frac{\delta v}{\delta x}=2x \quad \frac{\delta v}{\delta y}=2y$$

Also, $(u,v)=(4,5)$ when $(x,y)=(2,1)$.

Calculate the rate of change of temperature as *y* changes when $(x,y)=(2,1)$.

(A) $6e^{20}$

(B) $12e^{20}$

(C) $20e^{20}$

(D) $28e^{20}$

(E) $54e^{20}$

Question 7.2 *Reference: BPP*

The temperature of a particle located at the point (x,y) is $T=f(x,y)=e^{0.01x^2+0.04y^2}$.

A particle is located at the point $(10,5)$.

In which direction should the particle depart this point in order to experience the maximum possible rate of cooling?

(A) $\left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$

(B) $\left\langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$

(C) $\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

(D) $\left\langle \frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

(E) $\left\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$

Practice questions

Question 7.3

Reference: November 2001, Question 3

Sales, *S*, of a new insurance product are dependent on the labor, *L*, of the sales force and the amount of advertising, *A*, for the product.

This relationship can be modeled by:

$$S = 175L^{1.5}A^{0.8}$$

Which of the following statements is true?

(A) *S* increases at an increasing rate as *L* increases and increases at a decreasing rate as *A* increases.
(B) *S* increases at an increasing rate as *L* increases and increases at an increasing rate as *A* increases.
(C) *S* increases at a decreasing rate as *L* increases and increases at a decreasing rate as *A* increases.
(D) *S* increases at a decreasing rate as *L* increases and increases at an increasing rate as *A* increases.
(E) *S* increases at a constant rate as *L* increases and increases at a constant rate as *A* increases.

Question 7.4

Reference: BPP

Suppose that $z = f(x,y) = x^2 + 2y^2 + 3xy - 2x + 5$.

Which of the following statements about local extreme points is correct?

(A) There is a local maximum at the point $(-8,6)$.
(B) There is a local minimum at the point $(-8,6)$.
(C) There is a local maximum at the point $(8,-6)$.
(D) There is a local minimum at the point $(8,-6)$.
(E) The function has no local extreme points.

Practice questions

Question 7.5

Reference: May 2000, Question 30

Three radio antennas are located at points $(1,2)$, $(3,0)$, $(4,4)$ in the *xy*-plane.

In order to minimize static, a transmitter should be located at the point that minimizes the sum of the weighted squared distances between the transmitter and each of the antennas.

The weights are 5, 10, and 15 respectively, for the three antennas.

What is the *x*-coordinate of the point at which the transmitter should be located in order to minimize the static?

(A) 2.67
(B) 3.17
(C) 3.33
(D) 3.50
(E) 4.00

Question 7.6

Reference: BPP

Determine the absolute maximum value of $f(x,y)=12+3x+4y$ on the closed and bounded triangle whose vertices are $(-1,2)$, $(1,1)$, $(3,-1)$.

(A) 11
(B) 13
(C) 15
(D) 17
(E) 19

