Course 1 Key Concepts

Calculus

Lesson 8

Multivariable integral calculus

Overview

In this final calculus lesson, we'll describe how to calculate the volume of certain solid figures in 3-space by evaluating a double integral. We'll also look at ways to simplify the algebra, including the use of polar coordinates and changing the order of integration.

We'll conclude the lesson by studying how to calculate the average value of a function of two variables and the "average location" of a region in the *xy*-plane. These techniques have important applications in probability theory to calculate joint probabilities, moments of a function of two random variables, as well as the covariance between two random variables

BPP Learning Objectives

This lesson covers the following BPP learning objectives:

(C24) *Identify and evaluate a double integral equal to the volume of a solid figure that is under a surface $z = f(x, y)$ and above a region D in the xy-plane.*

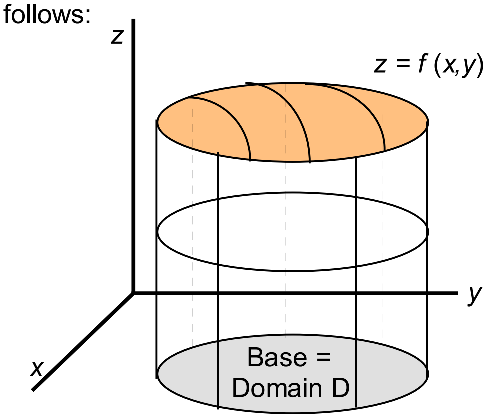
(C25) *Use double integration to find the average value of a function of two variables and the centroid of a region in the plane.*

Theory

Solid figures in 3-space

We begin by describing a solid figure in 3-space that is related to the function $z = f(x,y)$. For the sake of simplicity we'll assume throughout this section that this function is non-negative.

The solid figure *S* that is associated with this function is described as follows:



- The base of the solid *S* is the domain *D* in the *xy*-plane of the function $z = f(x,y)$.
- The top of the solid *S* is the surface formed by the graph of the function $z = f(x,y)$.
- The side walls of the solid *S* are lines perpendicular to the *xy*-plane from the edge of the domain to the edge of the top surface.

The solid *S* is shown in the figure on the right.

We want to focus on the calculation of the volume of this solid figure.

You will often see the following double integral symbols employed as standard notation for the volume of the solid figure *S*:

$$\iint_D f(x,y)\,dA \qquad \text{or} \qquad \iint_D f(x,y)\,dy\,dx$$

Just as in single-variable calculus, there is a theoretical definition of $\iint_D f(x,y)\,dA$ as a limit of a Riemann sum.

The base of the solid figure *S* is sub-divided by a rectangular grid. For each rectangle R_{ij} in the grid that touches *D*, a point (x_{ij}, y_{ij}) in the rectangle (which is also in *D*) is chosen and then $z_{ij} = f(x_{ij}, y_{ij})$ is used as the height of a rectangular solid. The volume of *S* is then approximated by the Riemann sum

$$\sum_i \sum_j \underbrace{Area(R_{ij}) \times f(x_{ij}, y_{ij})}_{\text{volume of a rectangular solid}}$$

of volumes of rectangular solids.

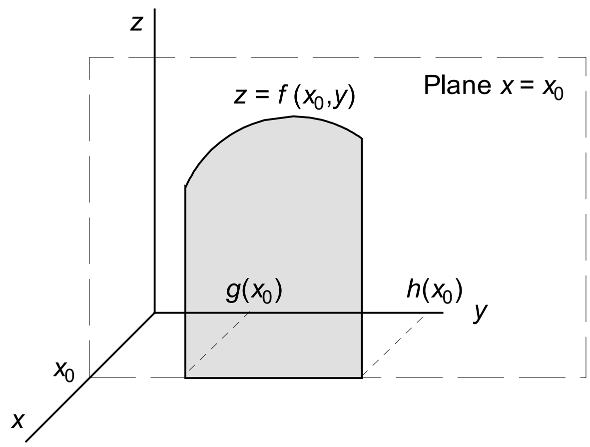
The precise volume of *S* is defined to be the limit of a Riemann sum as the mesh of the grid sub-dividing the base approaches zero. All of the basic properties of double integrals are then derived from this definition as a limit of a Riemann sum.

Theory

Fubini's Theorem

Continuing the discussion of the volume of the solid figure S described above, Fubini's Theorem provides a method for the calculation of the volume of S in terms of a sequence of two single-variable integrals (known as an **iterated integral**).

Consider the solid figure S and think about cutting it into very thin parallel slices.



When the solid S is intersected ("sliced") by the plane $X = x_0$, we see a 2-dimensional cross-sectional region.

We'll define the boundary of the domain of the function as $g(x_0)$ and $h(x_0)$, where $g(x_0) < h(x_0)$.

It should be clear that we can use our single-variable calculus techniques here to calculate the "area under the curve" of $z = f(x_0, y)$ over the interval $[g(x_0), h(x_0)]$.

The volume of this cross sectional solid will then be equal to the area under the curve multiplied by the thickness of the cross section.

The area of this cross sectional region is denoted by $A(x_0)$, and is calculated as:

$$A(x_0) = \int_{y=g(x_0)}^{h(x_0)} f(x_0, y)\,dy$$

If we now take a cross section of S by a general plane $X = x$ and give the cross-sectional region a small "thickness" dx. This process results in a "thin slice" of the solid figure whose volume is approximately $A(x)dx$. The total volume of S should be the sum of these thin slices of volume. This is essentially how Fubini's Theorem for the calculation of the volume of S is derived.

Fubini's Theorem

Let solid, S, have a base equal to the domain of $z = f(x, y)$, a set D described by:

$$a \le x \le b,\ g(x) \le y \le h(x).$$

If the top of the solid figure S is the surface $z = f(x, y)$ and the side-walls of the solid S are lines perpendicular to the xy-plane from the edge of the domain to the edge of the top surface, then:

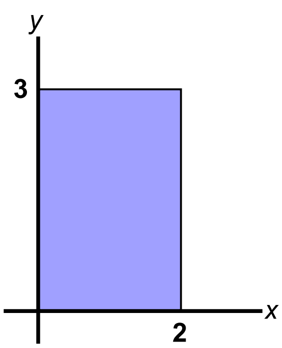
$$\text{Volume}(S) = \int_{x=a}^{b} \underbrace{A(x)dx}_{\text{volume of a thin slice}} = \underbrace{\int_{x=a}^{b}\left(\int_{y=g(x)}^{h(x)} f(x,y)\,dy\right)dx}_{\text{iterated integral}}$$

Theory

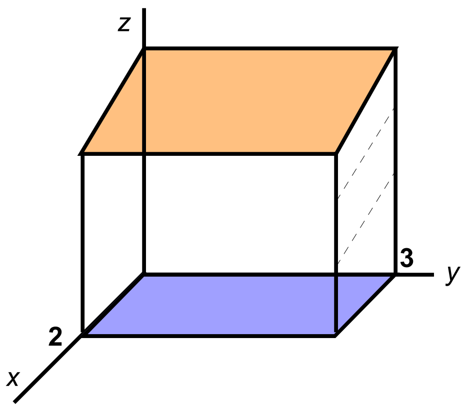
An example of Fubini's Theorem

Let's see how Fubini's Theorem is used to calculate the volume of a simple solid figure. Suppose that we take the rectangle $0 \le x \le 2$, $0 \le y \le 3$ as the domain (*ie* the base D of the solid S) of the function $z = 5 - x$ (a plane). The solid figure under this surface and above D is picture below.

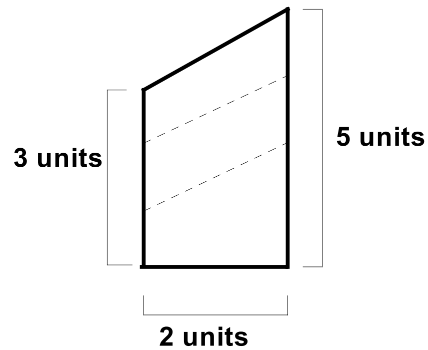
Base



The Solid Figure S



Right End Trapezoid



According to Fubini's Theorem we should be able to calculate the volume of this solid figure as:

$$\text{Volume}(S) = \iint_D (5-x)\,dy\,dx = \int_{x=0}^{2} \left(\int_{y=0}^{3} (5-x)\,dy \right) dx$$

since the base region is bounded on the left by $x = a = 0$, on the right by $x = b = 2$, on the lower side by the graph of $y = g(x) = 0$, and on the upper side by the graph of $y = h(x) = 3$, and the top surface is the plane $z = 5 - x$.

We evaluate the "inside" integral $\int_{y=0}^{3} (5-x)\,dy$ by thinking of x as being fixed and y as varying.

So, we have:

$$\text{Volume}(S) = \int_{x=0}^{2} \left(\int_{y=0}^{3} (5-x)\,dy \right) dx = \int_{x=0}^{2} \left((5-x)\,y \Big|_{y=0}^{y=3} \right) dx$$

$$= \int_{x=0}^{2} \underbrace{(5-x)\,3\,dx}_{A(x)} = \left(15x - \frac{3x^2}{2} \right) \Bigg|_0^2 = 30 - \frac{12}{2} = 24$$

Theory

Properties of double integrals

When $f(x,y)=1$, the solid figure S described earlier is a rectangular solid with height 1 and base area equal to $\text{Area}(D)$. So in this case we have $\text{Volume}(S)=\text{Area}(D)\times\text{height}=\text{Area}(D)\times 1$.

The second property below results from the basic definition of the double integral as a limit of a Riemann sum.

Properties of double integrals

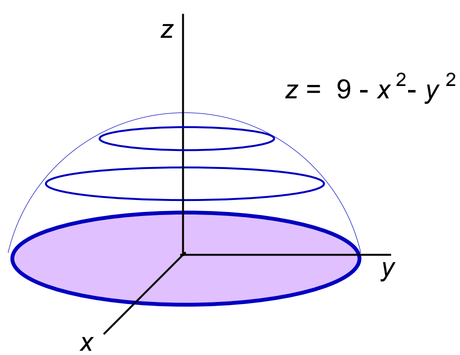
(i) $\displaystyle\iint_D 1\,dA=\text{Area}(D)$

(ii) $\displaystyle\iint_D \left(af(x,y)+bg(x,y)+c\right)dA=a\iint_D f(x,y)\,dA+b\iint_D g(x,y)\,dA+c\times\text{Area}(D)$

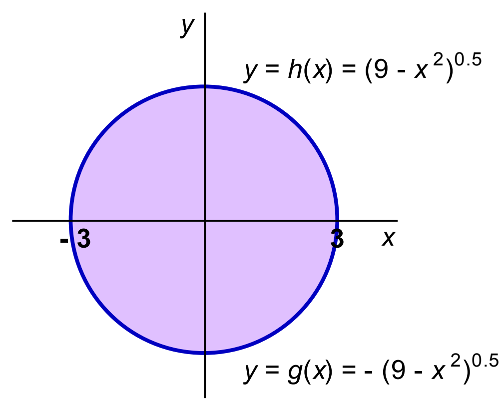
Let's take a look at another example of Fubini's Theorem.

Let's calculate the volume beneath the surface $z=9-x^2-y^2$ and above the circular region $x^2+y^2\le 9$ in the *xy*-plane. Notice that this surface intersects the *xy*-plane in the circle $x^2+y^2=9$.

Solid figure S



Base region D



The boundary curve of the base region is the circle $x^2+y^2=9$. This equation must be solved for *y* in terms of *x* to find boundary curves as functions of *x*:

$$x^2+y^2=9 \quad\Rightarrow\quad y=\pm\sqrt{9-x^2}$$

The upper boundary curve is the semi-circle $y=h(x)=\sqrt{9-x^2}$ and the lower boundary curve is the semi-circle $y=g(x)=-\sqrt{9-x^2}$.

Theory

The left and right extremes of the base region are $x = a = -3$ and $x = b = 3$ respectively. So by Fubini's Theorem, the volume of S is given by the iterated integral:

$$\text{Volume}(S) = \int_{x=-3}^{3} \left(\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 9 - x^2 - y^2 \, dy \right) dx$$

$$= \int_{x=-3}^{3} \left((9-x^2)y - \frac{y^3}{3} \right) \Bigg|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx = \int_{x=-3}^{3} \frac{4}{3}\left(9-x^2\right)^{3/2} dx$$

The final integration can be done with a trignometric substitution $x = 3\sin(\theta)$, resulting in a volume of $81\pi/2$.

An alternative method is to use polar coordinates – let's look at this next.

Double integrals in polar coordinates

Theorem

If the region D described in rectangular coordinates by $a \le x \le b$, $g(x) \le y \le h(x)$ can be described in polar coordinates by $\alpha \le \theta \le \beta$, $j(\theta) \le r \le k(\theta)$, then we have:

$$\iint_D f(x,y)\,dy\,dx = \int_{x=a}^{b} \left(\int_{y=g(x)}^{h(x)} f(x,y)\,dy \right) dx = \int_{\theta=\alpha}^{\beta} \left(\int_{r=j(\theta)}^{k(\theta)} f\left(r\cos(\theta), r\sin(\theta)\right) r\,dr \right) d\theta$$

Now let's return to the calculation of the volume that lies beneath the surface $z = 9 - x^2 - y^2$ and above the circular region $x^2 + y^2 \le 9$ in the xy-plane. This circular region can be described in polar coordinates by $0 \le \theta \le 2\pi$, $0 \le r \le 3$. Also, we have:

$$f\left(r\cos(\theta), r\sin(\theta)\right) = 9 - \left(r\cos(\theta)\right)^2 - \left(r\sin(\theta)\right)^2 = 9 - r^2$$

So by the theorem above we have:

$$\text{Volume}(S) = \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{3} \left(9 - r^2\right) r\,dr \right) d\theta = \int_{\theta=0}^{2\pi} \left(\left(\frac{9r^2}{2} - \frac{r^4}{4} \right) \Bigg|_{r=0}^{3} \right) d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{81}{4} \right) d\theta = \frac{81}{4} \times 2\pi = \frac{81\pi}{2}$$

In this case, the polar conversion method is much easier than integrating in rectangular coordinates.

Theory

Changing the order of integration

When we calculate the volume of a solid figure S that is above D and beneath the surface $z = f(x,y)$ via the iterated integral:

$$\text{Volume}(S) = \int_{x=a}^{b} \left(\int_{y=g(x)}^{h(x)} f(x,y)\,dy \right) dx$$

we are slicing the solid figure by planes $X = x$ where $a \le x \le b$. The inside integral represents the area of a cross section of S by $X = x$. These planes are perpendicular to the x-axis. They could also be described as being planes parallel to the yz-plane. If evaluation of this iterated integral proves troublesome, you might attempt to slice the solid by planes parallel to the y-axis to convert the volume calculation to an iterated integral where the inside integration is with respect to x. Let's look at the following example to illustrate this idea.

Consider the iterated integral given by:

$$\int_{x=0}^{2} \left(\int_{y=x}^{2} e^{y^2}\,dy \right) dx$$

The inside integral presents a problem: there is no closed-form expression for the anti-derivative $\int e^{y^2}\,dy$.

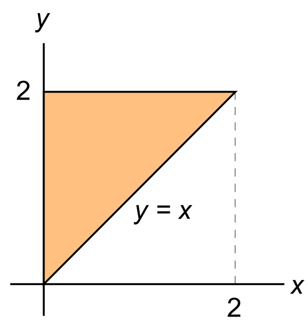
So let's consider the domain associated with this volume calculation and see if the problem becomes simpler if we slice the solid by planes perpendicular to the y-axis. The domain D associated with the iterated integral is the triangle pictured on the right.

When we slice the solid figure S by planes of the type $X = x$, then the points in the triangle are described by:

$$0 \le x \le 2,\ x \le y \le 2$$

When we slice by planes of the type $Y = y$, then the points in the triangle are described by:

$$0 \le y \le 2,\ 0 \le x \le y$$



We can now attempt the iterated integration as follows:

$$\int_{x=0}^{2} \left(\int_{y=x}^{2} e^{y^2}\,dy \right) dx = \int_{y=0}^{2} \left(\int_{x=0}^{y} e^{y^2}\,dx \right) dy = \int_{y=0}^{2} e^{y^2} \left(\int_{x=0}^{y} 1\,dx \right) dy$$

$$= \int_{y=0}^{2} e^{y^2}(y)\,dy = \left(\frac{e^{y^2}}{2} \right)\Bigg|_{y=0}^{2} = \frac{e^4 - 1}{2}$$

Theory

Centroid of a region D in the xy-plane

Definition

If D is a region in the plane $\mathbb{R}^2$, then the **centroid** of D is the point $(\overline{x},\overline{y})$ defined by:

$$\overline{x} = \frac{\iint_D x\,dy\,dx}{\iint_D 1\;dy\,dx} = \frac{\iint_D x\,dy\,dx}{\text{Area}(D)} \qquad \overline{y} = \frac{\iint_D y\,dy\,dx}{\iint_D 1\;dy\,dx} = \frac{\iint_D y\,dy\,dx}{\text{Area}(D)}$$

You can think of the centroid as the center (or "average location") of x and y over the region.

For example, suppose that D is the triangular region $0 \le x \le 2$, $0 \le y \le x$.

Then we would calculate the centroid of D as follows:

$$\overline{x} = \frac{\iint_D x\,dy\,dx}{\text{Area}(D)} = \frac{\int_{x=0}^{2}\left(\int_{y=0}^{x} x\,dy\right)dx}{\frac{1}{2}\times 2\times 2} = \frac{\int_{x=0}^{2}\left(xy\Big|_{y=0}^{x}\right)dx}{2} = \frac{\int_{x=0}^{2} x^2\,dx}{2} = \frac{\left.\frac{x^3}{3}\right|_{x=0}^{2}}{2} = \frac{8}{6} = \frac{4}{3}$$

$$\overline{y} = \frac{\iint_D y\,dy\,dx}{\text{Area}(D)} = \frac{\int_{x=0}^{2}\left(\int_{y=0}^{x} y\,dy\right)dx}{\frac{1}{2}\times 2\times 2} = \frac{\int_{x=0}^{2}\left(\left.\frac{y^2}{2}\right|_{y=0}^{x}\right)dx}{2} = \frac{\int_{x=0}^{2} \frac{x^2}{2}\,dx}{2} = \frac{\left.\frac{x^3}{6}\right|_{x=0}^{2}}{2} = \frac{2}{3}$$

Hence, the centroid is $\left(\frac{4}{3},\frac{2}{3}\right)$.

Does this make sense intuitively? Well, there is theorem in geometry that states that the three lines from vertices to midpoints of the opposite sides of a triangle intersect at a point that is two-thirds of the way from any vertex to the opposite side. It turns out that this intersection point is the centroid $(\overline{x},\overline{y})$ computed above.

To see this, consider the line drawn from the vertex at $(0,0)$ to the midpoint at $(2,1)$. The point two-thirds along this line from the vertex has coordinates:

$$x = \frac{2}{3}\times(2-0) = \frac{4}{3} \qquad y = \frac{2}{3}\times(1-0) = \frac{2}{3}$$

Theory

Average value of a function of 2 variables

Definition

If the function $z = f(x,y)$ is defined on a region D in the plane $\mathbb{R}^2$, then the **average value** of $f(x,y)$ over D can be calculated by:

$$\overline{f(x,y)} = \frac{\iint_D f(x,y)\,dy\,dx}{\iint_D 1\ dy\,dx} = \frac{\iint_D f(x,y)\,dy\,dx}{\text{Area}(D)}$$

For example, suppose that $f(x,y) = x + y$ and D is the rectangular region $0 \le x \le 2$, $0 \le y \le 2$.

Then the average value is:

$$\overline{f(x,y)} = \frac{\iint_D f(x,y)\,dy\,dx}{\text{Area}(D)} = \frac{\int_{x=0}^{2}\left(\int_{y=0}^{2}(x+y)\,dy\right)dx}{4} = \frac{\int_{x=0}^{2}\left(xy + \left.\frac{y^2}{2}\right|_{y=0}^{2}\right)dx}{4}$$

$$= \frac{\int_{x=0}^{2}(2x+2)\,dx}{4} = \frac{\left.\left(x^2+2x\right)\right|_{x=0}^{2}}{4} = \frac{4+4}{4} = 2$$

Does this make sense intuitively?

The function varies linearly over a region that is rectangular. The centroid ("average location") of this region is the point $(1, 1)$ where the diagonals cross. Due to linearity, we might suspect that the average function value is the function value at the "average location." In fact, we have $f(1,1) = 1 + 1 = 2$.

If you now look back at the definition of the centroid of a region D in light of what we have seen here, you will notice that $\bar{x}$ is the average value over D of the function $f(x,y) = x$, and $\bar{y}$ is the average value over D of the function $f(x,y) = y$.

Theory

Triple integrals

You may see iterated triple integrals of the form:

$$\int_{x=a}^{b}\left(\int_{y=g(x)}^{h(x)}\left(\int_{z=k(x,y)}^{f(x,y)} p(x,y,z)\,dz\right)dy\right)dx$$

In the inside integral, z is varying and both x and y are assumed to be constant. So the first step is to find a function $P(x,y,z)$ whose partial derivative with respect to z is the function $p(x,y,z)$.

Formally:

$$P_z(x,y,z) = p(x,y,z) \Rightarrow \int_{z=k(x,y)}^{f(x,y)} p(x,y,z)\,dz = P(x,y,f(x,y)) - P(x,y,k(x,y))$$

$$\Rightarrow \int_{x=a}^{b}\left(\int_{y=g(x)}^{h(x)}\left(\int_{z=k(x,y)}^{f(x,y)} p(x,y,z)\,dz\right)dy\right)dx = \int_{x=a}^{b}\left(\int_{y=g(x)}^{h(x)}\big(P(x,y,f(x,y)) - P(x,y,k(x,y))\big)dy\right)dx$$

When the function $p(x,y,z)=1$ and the surface $z=f(x,y)$ is always above the surface $z=k(x,y)$, then the iterated triple integral given above can be interpreted as the volume of a solid figure that is in between these two surfaces and above the region D in the xy-plane. However, it is really not necessary to use a triple integral for this volume. A special case of this situation is when the “lower” surface is $z=k(x,y)=0$. The resulting solid figure is the one we studied earlier.

Iterated triple integrals might also be used to calculate the **centroid** $(\bar{x},\bar{y},\bar{z})$ of a solid 3-dimensional figure S:

$$\bar{x} = \frac{\iiint_S x\,dz\,dy\,dx}{\text{Volume}(S)} \qquad \bar{y} = \frac{\iiint_S y\,dz\,dy\,dx}{\text{Volume}(S)} \qquad \bar{z} = \frac{\iiint_S z\,dz\,dy\,dx}{\text{Volume}(S)}$$

Worked examples

Example 8.1

Reference: BPP

Find the volume of the tetrahedron whose surfaces are the planes $x = 0$, $y = 0$, $z = 0$ and $x + 2y + 3z = 12$.

(A) 24
(B) 32
(C) 48
(D) 56
(E) 64

Solution

The tetrahedron is a solid figure *S* that is beneath the surface:

$$z = \frac{12 - x - 2y}{3}$$

and above the triangular region in the *xy*-plane given by:

$$0 \le x \le 12, \quad 0 \le y \le \frac{12 - x}{2}.$$

So the volume can be calculated from the iterated double integral:

$$\text{Volume} = \int_{x=0}^{12} \left(\int_{y=0}^{(12-x)/2} \frac{12 - x - 2y}{3}\, dy \right) dx$$

$$= \int_{x=0}^{12} \left(\left(\frac{(12-x)y}{3} - \frac{y^2}{3} \right) \Bigg|_{y=0}^{(12-x)/2} \right) dx$$

$$= \int_{x=0}^{12} \left(\frac{(12-x)^2}{6} - \frac{(12-x)^2}{12} \right) dx = \int_{x=0}^{12} \frac{(12-x)^2}{12} dx$$

$$= -\frac{(12-x)^3}{36} \Bigg|_{x=0}^{12} = \frac{12^3}{36} = 48$$

So, the correct answer is **C**.

Worked examples

Example 8.2

Reference: BPP

Evaluate $\int_{x=0}^{2}\left(\int_{y=x}^{2}\sqrt{12+y^2}\,dy\right)dx$.

(A) $\frac{64}{3}-4\sqrt{12}$

(B) $\frac{64}{3}-2\sqrt{12}$

(C) $12\sqrt{12}-\frac{64}{3}$

(D) $64-12\sqrt{12}$

(E) $64-4\sqrt{12}$

Solution

The inside integration would require a messy trig substitution. So, let's reverse the order of integration:

$$\int_{x=0}^{2}\left(\int_{y=x}^{2}\sqrt{12+y^2}\,dy\right)dx=\int_{y=0}^{2}\left(\int_{x=0}^{y}\sqrt{12+y^2}\,dx\right)dy$$

$$=\int_{y=0}^{2}\sqrt{12+y^2}\cdot y\,dy=\left.\frac{\left(12+y^2\right)^{3/2}}{3}\right|_{y=0}^{2}$$

$$=\frac{16^{3/2}-12^{3/2}}{3}=\frac{64-12\sqrt{12}}{3}$$

$$=\frac{64}{3}-4\sqrt{12}$$

So, the correct answer is **A**.

Worked examples

Example 8.3

Reference: BPP

Find the *y* coordinate of the centroid of the semi-circle given by $-5 \le x \le 5$, $0 \le y \le \sqrt{25 - x^2}$.

(A) $\dfrac{1}{\pi}$

(B) $\dfrac{2\pi}{3}$

(C) π

(D) $\dfrac{20}{3\pi}$

(E) $\dfrac{10}{\pi}$

Solution

The *y*-coordinate of the centroid is calculated as:

$$\bar{y} = \frac{\iint\limits_D y\,dy\,dx}{\text{Area}(D)} = \frac{\int\limits_{x=-5}^{5}\left(\int\limits_{y=0}^{\sqrt{25-x^2}} y\,dy\right)dx}{\pi 5^2/2}$$

$$= \frac{\int\limits_{x=-5}^{5}\left(\frac{25-x^2}{2}\right)dx}{\pi 5^2/2} = \left.\frac{25x - x^3/3}{25\pi}\right|_{-5}^{5} = \frac{20}{3\pi}$$

So, the correct answer is **D**.

Practice questions

Question 8.1

Reference: May 2000, Question 31

Let *R* be the region bounded by the graph of $x^2 + y^2 = 9$.

Calculate $\iint_R \left(x^2 + y^2 + 1\right) dA$.

(A) 24π

(B) $\dfrac{99\pi}{4}$

(C) $\dfrac{81\pi}{2}$

(D) $\dfrac{99\pi}{2}$

(E) $\dfrac{6723\pi}{4}$

Question 8.2

Reference: November 2000, Question 10

Let *S* be a solid in 3-space and *f* a function defined on *S* such that:

$$\iiint_S f(x,y,z)\,dV = 5$$

and

$$\iiint_S \left(4f(x,y,z) + 3\right) dV = 47$$

Calculate the volume of *S*.

(A) 2
(B) 5
(C) 7
(D) 9
(E) 14

Practice questions

Question 8.3

Reference: May 2001, Question 36

A town in the shape of a square with each side measuring 4 units has an industrial plant at its center. The industrial plant is polluting the air such that the concentration of pollutants at each location (x, y) in the town can be modeled by the function:

$$C(x,y) = 22{,}500\left(8 - x^2 - y^2\right) \text{ for } -2 \le x \le 2 \text{ and } -2 \le y \le 2.$$

Calculate the average pollution concentration over the entire town.

(A) 30,000
(B) 120,000
(C) 480,000
(D) 1,920,000
(E) 7,680,000

Question 8.4

Reference: November 2001, Question 2

Let R be a region in the *xy*-plane with area 2. Let $\iint_R f(x,y)\,dA = 6$.

Determine $\iint_R \left[4f(x,y) - 2\right] dA$.

(A) 12
(B) 18
(C) 20
(D) 22
(E) 44

Question 8.5

Reference: Sample Exam, Question 6

Calculate $\int_0^{\infty}\left(\int_0^{x}\left(1 + x^2 + y^2\right)^{-2} dy\right) dx$.

(A) 0
(B) $\frac{\pi}{16}$
(C) $\frac{\pi}{8}$
(D) $\frac{\pi}{4}$
(E) π