Solutions to practice questions

Calculus Lessons 1 – 8

Calculus Lesson 1

Solution 1.1

Reference: BPP, Answer C

Looking at the statements in turn:

I. The right-hand limit is: $\lim_{x\to 1^+} \frac{x^2+1}{2x-2} = \infty$ since the numerator is near 2 and the denominator is small and positive if x is slightly greater than 1.

The left-hand limit is: $\lim_{x\to 1^-} \frac{x^2+1}{2x-2} = -\infty$ since the numerator is near 2 and the denominator is small and negative if x is slightly less than 1.

So, statement **I** is incorrect.

II. For $x \neq 1$ we have $\frac{x^2-x}{x^2-2x+1} = \frac{x}{x-1}$. For x slightly greater than 1 the numerator is near 1 and the denominator is small and positive. Hence, we have:

$$\lim_{x\to 1^+} \frac{x^2-x}{x^2-2x+1} = \infty$$

So, statement **II** is incorrect.

III. Factoring $|x|^{1.5}$ out of the radical and canceling, we obtain the following relation for $x<0$:

$$\frac{\sqrt{|x|^{1.5}+10{,}000}}{x} = \frac{|x|^{.75}\sqrt{1+10{,}000/|x|^{1.5}}}{-|x|} = \frac{\sqrt{1+10{,}000/|x|^{1.5}}}{-|x|^{.25}}$$

In the final form the numerator approaches 1 as $x \to -\infty$ and the denominator becomes large and negative. So the ratio is near zero (on the negative side).

So, statement **III** is correct.

So, the correct answer is **C**.

Solution 1.2

Reference: BPP, Answer D

This is an important conceptual question.

Let's start by considering the case when $f(x) = cx$ and $g(x) = x$.

Then for $a = 0$, we have:

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

Now:

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{cx}{x} = \lim_{x \to 0} c = c$$

This immediately rules out answers A, B and C.

Is it possible that the limit does not exist? Certainly.

For example, if $f(x) = x$ and $g(x) = x^2$, then for $a = 0$ we have:

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{x}{x^2} = \lim_{x \to 0^+} \frac{1}{x} = \infty$$

So, statement D is correct.

Finally, statement E considers only two points. This tells us nothing about the true limit as x approaches a.

So, the correct answer is **D**.

Solution 1.3

Reference: BPP, Answer D

The function is continuous at $x = 5{,}000$ since the left and right-hand limits at this point are both equal to 0.

The only possible discontinuity is at $x = 20{,}000$.

The left-hand limit at 20,000 is:

$$0.05(20{,}000) - 250 = 750$$

The right-hand limit is:

$$20{,}000r - c$$

For continuity it is thus necessary that the one-sided limits are equal. This requires the equation:

$$750 = 20{,}000r - c$$

And for the tax on 50,000 to be 3,000 we must also have the equation;

$$3{,}000 = 50{,}000r - c$$

Solving these two equations simultaneously results in $c = 750$ and $r = 0.075$.

So, the correct answer is **D**.

Solution 1.4

Reference: BPP, Answer A

Let's rewrite this spliced function as:

$$f(x)=\begin{cases} g(x) & \text{if } x<1 \\ h(x) & \text{if } x\geq 1 \end{cases}$$

where:

$$g(x)=\sqrt{(c-x)^2-1} \qquad\qquad h(x)=\frac{cx^2+x-2}{x^2-x+3}$$

The function $g(x)$ is an algebraic function that is defined at $x=1$ if:

$$|c-x|>1$$

Algebraic functions are continuous on their domains, so if $|c-x|>1$ then:

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} g(x)=\lim_{x\to 1} g(x)=g(1)=\sqrt{(c-1)^2-1}$$

Similarly, the function $h(x)$ is a rational function that is defined at $x=1$.

Rational functions are continuous on their domains, so:

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} h(x)=\lim_{x\to 1} h(x)=h(1)=\frac{c-1}{3}$$

To have continuity at $x=1$, the left-hand and right-hand limits must exist and be equal.

Setting the one-sided limits equal and solving for c:

$$\sqrt{(c-1)^2-1}=\frac{c-1}{3} \quad\Rightarrow\quad (c-1)^2-1=\frac{(c-1)^2}{9} \quad\Rightarrow\quad c=1\pm\sqrt{9/8}$$

Notice that both values of c satisfy the condition (for $x=1$):

$$|c-x|>1$$

However, the possible solution $c=1-\sqrt{9/8}$ is extraneous since this value of c results in a positive left-hand limit and a negative right-hand limit.

So, the only correct solution is $c=1+\sqrt{9/8}$

So, the correct answer is **A.**

Calculus Lesson 2

Solution 2.1

Reference: BPP, Answer C

Revenue is equal to the product of the price per unit by the number sold at this price, *ie*:

$$R(p) = p(2000 - 10p) = 2000p - 10p^2$$

Differentiate with respect to price to find the instantaneous rate of change:

$$R'(p) = 2000 - 20p$$
$$\Rightarrow R'(50) = 1000$$

So, the correct answer is **C**.

Solution 2.2

Reference: BPP, Answer B

We are asked to calculate $g'(\ln(2))$. The derivative can be calculated using the quotient rule as follows:

$$g'(t) = \frac{\left(1+Be^{-t}\right)(A)' - A\left(1+Be^{-t}\right)'}{\left(1+Be^{-t}\right)^2} = \frac{0 - A\left(-Be^{-t}\right)}{\left(1+Be^{-t}\right)^2} = \frac{ABe^{-t}}{\left(1+Be^{-t}\right)^2}$$

Recall that

$$e^{\ln(b)} = b \qquad \text{and} \qquad e^{-\ln(b)} = \frac{1}{b}$$

We have:

$$g'(\ln(2)) = \frac{ABe^{-\ln(2)}}{\left(1+Be^{-\ln(2)}\right)^2} = \frac{AB/2}{(1+B/2)^2} = \frac{2AB}{4+4B+B^2}$$

So, the correct answer is **B**.

Solution 2.3

Reference: BPP, Answer D

Revenue is $R(n) = nP(n)$ and profit is the difference between revenue and costs:

$$\text{Profit} = \text{Revenue} - \text{Cost} = nP(n) - C(n)$$
$$= n(100-n) - \left(n^2 + 4n + 100\right) = -100 + 96n - 2n^2$$

Differentiating, and setting the derivative to zero:

$$(\text{Profit})' = 96 - 4n$$
$$(\text{Profit})' = 0 \Rightarrow n = 24 \Rightarrow P(n) = 100 - n = 76$$

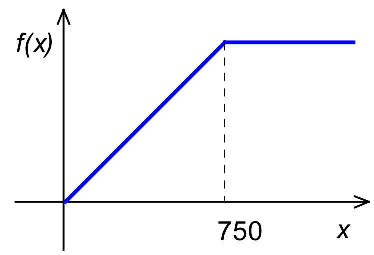
So, the correct answer is **D**.

Solution 2.4

Reference: May 2000, Question 39, Answer C

The formula for $f(x)$ is:

$$f(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x \le 750 \\ 750 & x > 750 \end{cases}$$



For $0 < x < 750$ the derivative is $(x)' = 1$.

For $x > 750$ the derivative is $(750)' = 0$.

So, the correct answer is **C**.

Note: Due to corners at $x = 0$ and $x = 750$, the derivative fails to exist at these points.

Solution 2.5

Reference: BPP, Answer E

The formula for $f(x)$ is:

$$f(x) = \begin{cases} g(x) = x^2 + x & \text{if } x \le 1 \\ h(x) = \dfrac{cx+d}{x+1} & \text{if } x > 1 \end{cases}$$

Since $g(x)$ is a polynomial function, it is differentiable (hence continuous) for all x. Also, $h(x)$ is a rational function that is defined for all $x \ne -1$. So it is differentiable (hence continuous) for all $x \ne -1$.

The spliced function $f(x)$ is continuous at the junction point $x = 1$ if:

$$g(1) = h(1) \Rightarrow 2 = \frac{c+d}{2}.$$

Now $g'(x) = 2x + 1$ and for the derivative of $h(x)$ we have from the quotient rule the following formula:

$$h'(x) = \frac{(x+1)(c) - (cx+d)(1)}{(x+1)^2} \quad \Rightarrow \quad h'(1) = \frac{c-d}{4}.$$

The spliced function is differentiable at $x = 1$ if we have:

$$g'(1) = h'(1) \Rightarrow 3 = \frac{c-d}{4}$$

Solving the simultaneous equations:

$$2 = \frac{c+d}{2} \qquad 3 = \frac{c-d}{4}$$

We find that:

$$c = 8, \; d = -4$$

So, the correct answer is **E**.

Calculus Lesson 3

Solution 3.1

Reference: November 2000, Question 8, Answer A

Let $n = 20 + x$ be the number of policies sold in a month.

Then the price at which they can be sold is $p = 40 - x$.

The cost of selling n policies is $C(n) = 100 + 32n$.

The profit is given by:

$$\begin{aligned}\text{Profit} &= \text{Revenue} - \text{Costs}\\ &= n\,p - (100 + 32n) = n(p - 32) - 100\\ &= (20 + x)(8 - x) - 100 = 60 - 12x - x^2\end{aligned}$$

Notice that we have a concave down parabola for the profit function.

The maximum occurs at the one critical point where the parabola is at its highest point:

$$0 = (\text{Profit})' = -12 - 2x \quad \Rightarrow \quad x = -6 \quad \Rightarrow \quad \text{Profit} = 60 - 12x - x^2 = 96$$

So, the correct answer is **A**.

Solution 3.2

Reference: November 2000, Question 24, Answer B

Let's start by using the given relation $f(x+h) - f(x) = 6xh + 3h^2$ to determine $f(2)$.

Setting $x = 1$, $h = 1$ and $f(1) = 5$, we have:

$$f(2) - f(1) = 6 \times 1 \times 1 + 3 \times 1^2 = 9 \Rightarrow f(2) = f(1) + 9 = 14$$

Next, we use the same relation to calculate $f'(2)$.

$$f(x+h) - f(x) = 6xh + 3h^2 \Rightarrow f(2+h) - f(2) = 12h + 3h^2$$

$$f'(2) = \lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{12h + 3h^2}{h} = \lim_{h \to 0} (12 + 3h) = 12$$

$$\Rightarrow f(2) - f'(2) = 14 - 12 = 2$$

So, the correct answer is **B**.

Solution 3.3

Reference: November 2001, Question 8, Answer D

We are asked to calculate the minimum value of $F(t) = t\,e^{-t}$ on the interval $[1,7]$.

The extreme values must occur at endpoints of the interval or at interior critical points:

$$F'(t) = e^{-t} - te^{-t} = e^{-t}(1-t)$$

Setting the derivative to zero:

$$F'(t) = 0$$
$$\Rightarrow e^{-t}(1-t) = 0$$
$$\Rightarrow t = 1$$

So there is no interior critical point.

We need only choose the smallest of the following:

$$F(1) = e^{-1} = 0.36788$$
$$F(7) = 7\,e^{-7} = 0.00638$$

The minimum is 0.0064 to 4 decimal places.

So, the correct answer is **D**.

Solution 3.4

Reference: May 2000, Question 26, Answer E

We know that $y(x)$ is increasing. We also know that $y(x)$ is concave down, because for any two points on the graph of y, the line segment joining those points lies entirely below the graph of y.

So, we must have $y' > 0$ and $y'' < 0$ for $0 \le x \le 30$.

Answer choices A and B are linear functions that are not concave down.

Answer choice C is concave up since $y'' = \dfrac{3k}{4\sqrt{x}} > 0$.

Answer choice D is also concave up since $y'' = 2k > 0$.

This leaves just Answer choice E. To check that this correct, the derivatives are:

$$y' = \frac{k}{x+1} > 0 \qquad \text{and} \qquad y'' = \frac{-k}{(x+1)^2} < 0$$

So, the correct answer is **E**.

Solution 3.5

Reference: November 2001, Question 14, Answer A

This question requires a visual application of L'Hopital's Rule. From the graph given in the question we can observe the following facts:

$\lim_{x\to 0} f(x) = 0, \ \lim_{x\to 0} g(x) = 0$ (both functions are continuous at zero)

$\lim_{x\to 0} f'(x) > 0$ (tangent line has a positive slope)

$\lim_{x\to 0} g'(x) < 0$ (tangent line has a negative slope)

Applying L'Hopitals Rule, we have:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} < 0$$

So, the correct answer is **A**.

Solution 3.6

Reference: November 2001, Question 36, Answer C

The price at which the policies are sold is $p = 60 + x$. Let $n = 80 - x$ be the number of policies sold in a month. The revenue per month is given by:

$$R(x) = np = (80 - x)(60 + x) = 4800 + 20x - x^2$$
$$\Rightarrow R'(x) = 20 - 2x$$

Setting the derivative to zero yields a critical point at $x = 10$. This is a maximum because $R''(x) < 0$.

The maximum monthly revenue is thus $R(10) = 4800 + 20 \times 10 - 10^2 = 4900$.

So, the correct answer is **C**.

Calculus Lesson 4

Solution 4.1

Reference: May 2000, Question 28, Answer C

This question is more conceptual than computational.

Let $P(t)$ be the price at time t. We are given $P(0)=100$ and $P'(t)=I(t)$.

From the FTC, we have:

$$P(t_1)-P(0)=\int_0^{t_1} P'(t)\,dt=\int_0^{t_1} I(t)\,dt \quad\Rightarrow\quad P(t_1)=100+\int_0^{t_1} I(t)\,dt$$

So $P(t_1)=100$ if $\int_0^{t_1} I(t)\,dt=0$.

Remember that area below the x-axis is negative in the definite integral calculation. So, from the figure in the question we can see that this integral is zero for some t_1 between 2 and 4.

So, the correct answer is **C**.

Solution 4.2

Reference: November 2000, Question 35, Answer A

Let $V(t)$ be the value at time t. We are given the following:

$$V'(t)=k\left(20-V(t)\right)\ ,\ 2=V(0)\ ,\ 3=V(1)$$

Move all terms involving the value $V(t)$ to the left side:

$$\frac{V'(t)}{20-V(t)}=k$$

Recall that:

$$\left(\ln\left(f(t)\right)\right)'=\frac{f'(t)}{f(t)} \quad\Rightarrow\quad -\left(\ln\left(20-V(t)\right)\right)'=\frac{V'(t)}{20-V(t)}$$

Integrating:

$$\int\frac{V'(t)}{20-V(t)}\,dt=\int k\,dt \quad\Rightarrow\quad -\ln\left(20-V(t)\right)=kt+c$$

$$\Rightarrow 20-V(t)=e^{-(kt+c)}=a\,e^{-kt} \quad \text{where } a=e^{-c}$$

$$\Rightarrow V(t)=e^{-(kt+c)}=20-a\,e^{-kt}$$

Determining the two unknown constants from the two given conditions:

$$V(0)=2 \Rightarrow 20-ae^{0}=2 \Rightarrow a=18$$

$$V(1)=3 \Rightarrow 20-18e^{-k}=3 \Rightarrow e^{-k}=17/18=0.94444$$

$$\Rightarrow V(3)=20-18e^{-3k}=20-18(0.94444)^3=4.84$$

So, the correct answer is **A**.

Solution 4.3

Reference: November 2001, Question 12, Answer E

Here is another question that requires no real computation, just some reasoning.

Since $f'(0)=0$, we have from the FTC the following result:

$$f'(x_1)-f'(0)=\int_0^{x_1} f''(x)dx \quad \Rightarrow \quad f'(x_1)=\int_0^{x_1} f''(x)dx$$

The integral on the right is area under the graph of the second derivative between 0 and x_1, where regions below the x-axis have "negative area."

From the graph of the second derivative in the question, we can see that this area is positive for all x between 0 and 5.

And if $f'>0$ on $[0,5]$, it follows that f is increasing on this interval.

So the maximum value must occur at the right endpoint $x=5$.

So, the correct answer is **E**.

Solution 4.4

Reference: November 2001, Question 26, Answer C

Let $s(t)$ be the sales by time t.

The instantaneous rate of changes of sales is the derivative $s'(t)$.

By the FTC, the total sales between times 2 and 4 are given by:

$$s(4)-s(2)=\int_2^4 s'(t)dt.$$

So if $s'(t)$ increases from $t+5/2$ to $t^2+1/2$, the increase in total sales over this time interval is:

$$\int_2^4 \left(t^2+\frac{1}{2}\right)-\left(t+\frac{5}{2}\right)dt=\int_2^4 t^2-t-2\ dt$$

$$=\left(\frac{t^3}{3}-\frac{t^2}{2}-2t\right)\Bigg|_2^4$$

$$=\frac{16}{3}-\left(-\frac{10}{3}\right)=\frac{26}{3}$$

So, the correct answer is **C**.

Solution 4.5

Reference: November 2001, Question 31, Answer E

Rearranging the equation:

$$\frac{y'(t)}{y(t)} = k_1 - k_2 \Rightarrow \left(\ln y(t)\right)' = k_1 - k_2$$

Integrating with respect to t:

$$\int \left(\ln y(t)\right)' dt = \int k_1 - k_2 \, dt$$

$$\Rightarrow \ln y(t) = t\left(k_1 - k_2\right) + c \quad \Rightarrow y(t) = ae^{t\left(k_1 - k_2\right)} \quad \text{where } a = e^c$$

Note that $y(0) = a$, so we can rewrite this as $y(t) = y(0)e^{t\left(k_1 - k_2\right)}$.

If there were no births (*ie* if $k_1 = 0$), the population would be halved at time $t = 8$.

So, we have:

$$e^{8\left(0 - k_2\right)} = \frac{1}{2} \quad \Rightarrow e^{8k_2} = 2 \quad \Rightarrow k_2 = \frac{\ln 2}{8}$$

So, the correct answer is **E**.

Solution 4.6

Reference: May 2001, Question 21, Answer E

The differential equation in this question is as complicated an equation as you are likely to meet in Course 1.

This question is easier if you notice that:

$$\frac{1}{Q\left(N - Q\right)} = \frac{1}{N}\left(\frac{1}{Q} + \frac{1}{N - Q}\right)$$

Then, rearranging the differential equation:

$$\frac{Q'(t)}{Q(t)\left(N - Q(t)\right)} = 1 \qquad \Rightarrow \frac{Q'(t)}{Q(t)} + \frac{Q'(t)}{N - Q(t)} = N \qquad \Rightarrow \left(\ln Q(t)\right)' - \left(\ln\left(N - Q(t)\right)\right)' = N$$

Integrating both sides of the equation with respect to t leads to:

$$\ln\left(Q\right) - \ln\left(N - Q\right) = Nt + c$$

$$\Rightarrow \ln\left(\frac{Q}{N - Q}\right) = Nt + c$$

$$\Rightarrow \frac{Q}{N - Q} = ae^{Nt} \qquad \text{where } a = e^c$$

$$\Rightarrow Q = \frac{aNe^{Nt}}{1 + ae^{Nt}}$$

So, the correct answer is **E**.

Calculus Lesson 5

Solution 5.1

Reference: November 2000, Question 29, Answer E

The number of losses reported in the first *n* years following the occurrence year is given by $R_n - R_0$. So the number of losses in all subsequent years is $\lim_{n\to\infty}(R_n - R_0) = \left(\lim_{n\to\infty} R_n\right) - R_0$. So the first step here is to derive a formula for R_n from the recursive definition given in the question:

$$\begin{aligned} R_n &= 2^{0.75^{n-1}} R_{n-1} = 2^{0.75^{n-1}}\left(2^{0.75^{n-2}} R_{n-2}\right) = \cdots \\ &= 2^{0.75^{n-1}} \times 2^{0.75^{n-2}} \times \cdots \times 2^{0.75^0} R_0 \\ &= 2^{\left(0.75^{n-1}+0.75^{n-2}+\cdots+0.75+1\right)} R_0 \end{aligned}$$

The exponent of 2 in the above relation is the partial sum of a convergent geometric series:

$$\lim_{n\to\infty} 0.75^{n-1} + 0.75^{n-2} + \cdots + 0.75 + 1 = \frac{1}{1-0.75} = 4 \qquad \text{(geometric series formula)}$$

$$\begin{aligned} \Rightarrow \left(\lim_{n\to\infty} R_n\right) - R_0 &= \left(\lim_{n\to\infty} 2^{\left(0.75^{n-1}+0.75^{n-2}+\cdots+0.75+1\right)} R_0\right) - R_0 \\ &= 2^4 R_0 - R_0 = 15 R_0 \end{aligned}$$

Since we are given $R_0 = 250$, it follows that $15R_0 = 3750$.

So, the correct answer is **E**.

Solution 5.2

Reference: November 2000, Question 39, Answer C

The total number of deaths in years 28 and after is given by:

$$\sum_{n=28}^{\infty} \frac{90{,}000}{(t+3)^3} = \frac{90{,}000}{31^3} + \frac{90{,}000}{32^3} + \cdots$$

This number cannot be computed precisely, but it can be estimated by using the same technique we employed to show that the harmonic series diverges. Since the function $f(t) = 1/t^3$ is a decreasing function of *t*, we have the following size estimates:

$$\frac{1}{n^3} = f(n) < \int_n^{n+1} \frac{1}{t^3}\,dt < f(n+1) = \frac{1}{(n+1)^3} \qquad \Rightarrow$$

$$\frac{90{,}000}{31^3} + \frac{90{,}000}{32^3} + \cdots < 90{,}000\left(\int_{31}^{32} \frac{1}{t^3}\,dt + \int_{32}^{33} \frac{1}{t^3}\,dt + \cdots\right) = \int_{28}^{\infty} \frac{90{,}000}{t^3}\,dt = \left.\frac{-90{,}000}{2t^2}\right|_{31}^{\infty} = 46.83$$

$$\frac{90{,}000}{31^3} + \frac{90{,}000}{32^3} + \cdots > 90{,}000\left(\int_{30}^{31} \frac{1}{t^3}\,dt + \int_{31}^{32} \frac{1}{t^3}\,dt + \cdots\right) = \int_{27}^{\infty} \frac{90{,}000}{t^3}\,dt = \left.\frac{-90{,}000}{2t^2}\right|_{30}^{\infty} = 50.00$$

So, the correct answer is **C**.

Solution 5.3

Reference: November 2001, Question 10, Answer E

This question requires knowledge about the n^{th} term test and p – series.

- The series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ diverges due to the n^{th} term test since the n^{th} term converges to 1.
- The series $\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{1}{n}\right)$ diverges since it is twice the divergent harmonic series.
- The series $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{1}{n}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{1}{n}$ diverges since $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a positive number.
- The series $\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}+\frac{1}{n}\right)$ is the same as $\frac{2}{2}+\frac{2}{4}+\frac{2}{6}+\cdots$. It diverges since it is twice the divergent harmonic series.
- The series $\sum_{n=1}^{\infty}\left(\frac{1-n}{n^2}+\frac{1}{n}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is the convergent p-series where $p=2$.

So, the correct answer is **E**.

Solution 5.4

Reference: May 2000, Question 29, Answer A

The prices vary according to the following:

$$P(4k)=100\times.9945^k\to 0 \quad \text{as } k\to\infty$$
$$P(4k+1)=100\times 1.30\times.9945^k\to 0 \quad \text{as } k\to\infty$$
$$P(4k+2)=100\times 1.30\times.85\times.9945^k\to 0 \quad \text{as } k\to\infty$$
$$P(4k+3)=100\times 1.30\times.85\times 1\times.9945^k\to 0 \quad \text{as } k\to\infty$$

This makes it clear that $P(n)\to 0$ as $n\to\infty$.

So, the correct answer is **A**.

Solution 5.5

Reference: May 2001, Question 2, Answer C

The n^{th} dividend is $D_n = 8 \times 1.07^{n-1}$.

The sum of the first *m* dividends is given by:

$$D_1 + \cdots + D_m = 8 \times \left(1 + 1.07 + \cdots + 1.07^{m-1}\right) = 8 \times \frac{1 - 1.07^m}{1 - 1.07} = 114.2857 \times \left(1.07^m - 1\right)$$

The sum of the dividends exceeds 500 if we have the following:

$$114.2857 \times \left(1.07^m - 1\right) > 500$$

$$\Rightarrow 1.07^m > 1 + \frac{500}{114.5827} = 5.375$$

$$\Rightarrow m > \frac{\ln(5.375)}{\ln(1.07)} > 24.86$$

$$\Rightarrow m \geq 25$$

So, the correct answer is **C**.

Solution 5.6

Reference: Sample Exam, Question 12, Answer C

In (iii) of Example 5.1 we saw the following:

$$\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a$$

From the formula given in the Question statement and the result in (iii) of Example 5.1, we saw that:

$$I = \lim_{n \to \infty} 100 \left(1 + \frac{c}{n}\right)^n = 100 \times \lim_{n \to \infty} \left(1 + \frac{c}{n}\right)^n = 100e^c$$

So, the correct answer is **C**.

Calculus Lesson 6

Solution 6.1

Reference: November 2000, Question 15, Answer C

First we find the values of t at each of the two given locations. It is easily seen from the parametric equations that the location is $(1,4,9)$ at time $t=1$ and $(16,32,36)$ at time $t=4$.

So, we have:

$$\mathbf{r}(t)=\left\langle t^2,4t^{3/2},9t\right\rangle$$
$$\Rightarrow \mathbf{v}(t)=\left\langle (t^2)',(4t^{3/2})',(9t)'\right\rangle=\left\langle 2t,6t^{1/2},9\right\rangle$$
$$\Rightarrow s'(t)=|\mathbf{v}(t)|=\sqrt{(2t)^2+\left(6t^{1/2}\right)^2+9^2}=\sqrt{4t^2+36t+81}=2t+9$$
$$\Rightarrow \text{Distance}=\int_1^4 s'(t)\,dt=\int_1^4 2t+9\,dt=\left(t^2+9t\right)\Big|_1^4=52-10=42$$

So, the correct answer is **C**.

Solution 6.2

Reference: SOA November 2000, Question 33, Answer E

When $\theta=0$, we have $r=2+\cos(0)=2+1=3$:

$$x=r\cos(0)=3\times 1=3\,,\ y=r\sin(0)=3\times 0=0\ \Rightarrow (x,y)=(3,0)$$

When $\theta=\pi$, we have $r=2+\cos(\pi)=2-1=1$:

$$x=r\cos(\pi)=1\times(-1)=-1\,,\ y=r\sin(\pi)=1\times 0=0\ \Rightarrow (x,y)=(-1,0)$$

When $\theta=\pi/3$, we have $r=2+\cos(\pi/3)=2+1/2=5/2$:

$$x=r\cos(\pi/3)=(5/2)\times(1/2)=5/4\,,\ y=r\sin(\pi/3)=(5/2)\times\left(\sqrt{3}/2\right)=5\sqrt{3}/4$$
$$\Rightarrow (x,y)=\left(5/4,5\sqrt{3}/4\right)$$

Two of the vertices lie along the horizontal axis. The distance between $(-1,0)$ and $(3,0)$ is 4.

So, the base of the triangle has length 4.

The height is the vertical distance from the point $\left(5/4,5\sqrt{3}/4\right)$ to the horizontal axis. So, the height of the triangle is $5\sqrt{3}/4$.

The area of the triangle is thus:

$$\frac{1}{2}\times\text{Base}\times\text{Height}=\frac{1}{2}\times 4\times\left(5\sqrt{3}/4\right)=5\sqrt{3}/2$$

So, the correct answer is **E**.

Solution 6.3

Reference: May 2001, Question 3, Answer C

The position vector is:

$$\mathbf{r}(t)=\left\langle 4\sin\frac{t}{2},\ 2t\cos t\right\rangle$$

The velocity vector is:

$$\mathbf{v}(t)=\mathbf{r}'(t)=\left\langle 2\cos\frac{t}{2},\ 2\cos t-2t\sin t\right\rangle$$

$$\Rightarrow \mathbf{v}\left(\frac{\pi}{2}\right)=\left\langle 2\cos\frac{\pi}{4},2\cos\frac{\pi}{2}-\pi\sin\frac{\pi}{2}\right\rangle=\left\langle 2\times\frac{1}{\sqrt{2}},0-\pi\right\rangle=\left\langle \sqrt{2},\ -\pi\right\rangle$$

Finally, the length of the velocity vector is:

$$\left|\mathbf{v}(\pi/2)\right|=\left|\left\langle \sqrt{2},-\pi\right\rangle\right|=\sqrt{2+\pi^2}$$

So, the correct answer is **C**.

Solution 6.4

Reference: May 2001, Question 15, Answer D

First, find the value of t at which the location is $(0,5)$.

To do this, solve the simultaneous equations:

$$0=2t^2+t-1 \quad \Rightarrow t=\frac{-1\pm\sqrt{9}}{4}\Rightarrow t=\frac{1}{2} \text{ or } t=-1$$

$$5=t^2-3t+1 \quad \Rightarrow t=\frac{3\pm\sqrt{25}}{2}\Rightarrow t=4 \text{ or } t=-1$$

Hence, $t=-1$.

The tangent slope to the curve is:

$$\frac{dy}{dx}=\frac{dy}{dt}\bigg/\frac{dx}{dt}=\frac{2t-3}{4t+1}$$

Hence, the tangent slope at $t=-1$ is:

$$\frac{dy}{dx}=\frac{2t-3}{4t+1}=\frac{-5}{-3}=\frac{5}{3}$$

So, the correct solution is **D**.

Solution 6.5

Reference: November 2001, Question 6, Answer E

The slope on the curve *C* at time *t* is given by:

$$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} = \frac{2t}{2t+1}$$

The formula of the line $5y - 4x = 3$ can be rewritten as:

$$y = \frac{4}{5}x + \frac{3}{5}$$

So, this line has slope $4/5$.

Setting these two slopes equal and solving for t, we have:

$$\frac{4}{5} = \frac{2t}{2t+1} \Rightarrow t = 2$$

So, the correct answer is **E**.

Solution 6.6

Reference: November 2001, Question 23, Answer C

Converting to rectangular coordinates:

$$r = \sin(\theta) + \sqrt{3}\cos(\theta)$$

$$\Rightarrow \sqrt{x^2 + y^2} = \frac{y}{\sqrt{x^2 + y^2}} + \frac{\sqrt{3}x}{\sqrt{x^2 + y^2}}$$

$$\Rightarrow x^2 + y^2 = y + \sqrt{3}x$$

The region to the left of $\theta = \pi/2$ is the region to the left of the *y*-axis.

The curve meets the *y*-axis when $x = 0$, *ie*:

$$y^2 = y \Rightarrow y = 0 \text{ or } y = 1$$

We have:

- $y = 1$ when $\theta = \pi/2$
- $y = 0$ when $0 = r = \sin(\theta) + \sqrt{3}\cos(\theta) \Rightarrow \theta = 2\pi/3$

So, the correct answer is **C**.

Calculus Lesson 7

Solution 7.1

Reference: May 2001, Question 18, Answer D

There are 2 paths from *y* to *T* through the intermediate variables *u* and *v*:

$$\frac{\delta T}{dy} = \frac{\delta u}{\delta y} \cdot \frac{\delta T}{\delta u} + \frac{\delta v}{\delta y} \cdot \frac{\delta T}{\delta v}$$

$$= (2x)\left(e^{uv}\, v\right) + (2y)\left(e^{uv}\, u\right)$$

$$= (4)\left(e^{20} \times 5\right) + (2)\left(e^{20} \times 4\right) = 28e^{20} \quad \text{when } (x,y) = (2,1),\ (u,v) = (4,5)$$

So, the correct answer is **D**.

Solution 7.2

Reference: BPP, Answer D

The greatest rate of increase is in the direction of the gradient. So the greatest rate of cooling is in the direction opposite to the direction of the gradient.

$$\nabla T(x,y) = \left\langle e^{0.01x^2+0.04y^2} \times 0.02x\,,\ e^{0.01x^2+0.04y^2} \times 0.08y \right\rangle \Rightarrow \nabla T(10,5) = e^2 \langle 0.2, 0.4 \rangle$$

Since the gradient is parallel to $\langle 1,2 \rangle$, a unit vector in the direction of the gradient is $\mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$.

A unit vector opposite to the gradient is $-\mathbf{u} = \left\langle \frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$.

So, the correct answer is **D**.

Solution 7.3

Reference: November 2001, Question 3, Answer A

The partial derivatives are:

$$\frac{\delta S}{\delta L} = 175 \times 1.5L^{0.5} \times A^{0.8} > 0 \qquad \frac{\delta^2 S}{\delta L^2} = 1.75 \times 1.5 \times 0.5L^{-0.5}\, A^{0.8} > 0$$

$$\frac{\delta S}{\delta A} = 175 \times L^{1.5} \times 0.8A^{-0.2} > 0 \qquad \frac{\delta^2 S}{\delta A^2} = 1.75 \times L^{1.5} \times 0.8 \times \left(-0.2A^{-1.2}\right) < 0$$

From the signs of the partial derivatives and their interpretations, the correct answer is **A**.

Solution 7.4

Reference: BPP, Answer E

For the function $z = f(x,y) = x^2 + 2y^2 + 3xy - 2x + 5$, we have:

$$f_x = 2x + 3y - 2 = 0 \quad f_y = 4y + 3x = 0 \quad \Rightarrow \quad (x,y) = (-8,6)$$

$$f_{xx} = 2 \quad f_{yy} = 4 \quad f_{xy} = 3 \quad \Rightarrow \quad \Delta f(x,y) = 2 \times 4 - 3^2 < 0$$

So there is a critical point, but the meaning of a negative discriminant is that this point is a saddle point, not a local extreme point.

So, the correct answer is **E**.

Solution 7.5

Reference: May 2000, Question 30, Answer B

The sum of weighted squared distances to minimize is:

$$z = 5\left((x-1)^2 + (y-2)^2\right) + 10\left((x-3)^2 + (y-0)^2\right) + 15\left((x-4)^2 + (y-4)^2\right)$$

$$= \underbrace{5(x-1)^2 + 10(x-3)^2 + 15(x-4)^2}_{g(x)} + \underbrace{5(y-2)^2 + 10(y-0)^2 + 15(y-4)^2}_{h(y)}$$

Because of this unusual separation of variables, the weighted sum of squared distances is minimized at the point (x_0, y_0) where $g(x)$ is minimized at x_0 and $h(y)$ is minimized at y_0.

$$g'(x) = 10(x-1) + 20(x-3) + 30(x-4) = 0 \quad \Rightarrow \quad x = \frac{190}{60} = 3.17$$

So, the correct answer is **B**.

Solution 7.6

Reference: BPP, Answer E

The equation is linear so its graph is a tilted segment of a plane above the given triangle. The highest point on this surface must be above one of the three vertices. So it suffices to plug the coordinates of the vertices into the function and then pick the largest of the tabulated function values:

$$f(x,y) = 12 + 3x + 4y \Rightarrow f(-1,2) = 17, \; f(1,1) = 19, \; f(3,-1) = 17$$

The maximum is 19 at $(1,1)$, so the correct answer is **E**.

Calculus Lesson 8

Solution 8.1

Reference: May 2000, Question 31, Answer D

A conversion to polar coordinates will simplify the calculation tremendously.

$$x^2+y^2+1=(r\cos\theta)^2+(r\sin\theta)^2+1=r^2+1$$

So, we have:

$$\iint_R\left(x^2+y^2+1\right)dA=\int_{\theta=0}^{2\pi}\left(\int_{r=0}^{3}\left(r^2+1\right)r\,dr\right)d\theta=\int_{\theta=0}^{2\pi}\left(\left(\frac{r^4}{4}+\frac{r^2}{2}\right)\Bigg|_{r=0}^{3}\right)d\theta$$

$$=\int_{\theta=0}^{2\pi}\left(\frac{81}{4}+\frac{9}{2}\right)d\theta=2\pi\left(\frac{81}{4}+\frac{9}{2}\right)=\frac{99\pi}{2}$$

So, the correct answer is **D**.

Solution 8.2

Reference: November 2000, Question 10, Answer D

We have:

$$\iiint_S\left(4f(x,y,z)+3\right)dV=4\underbrace{\iiint_S f(x,y,z)\,dV}_{5\text{ (given)}}+3\underbrace{\iiint_S 1\,dV}_{\text{Volume}(S)}=4\times5+3\times\text{Volume}(S)$$

$$\Rightarrow 47=20+3\times\text{Volume}(S)$$

$$\Rightarrow \text{Volume}(S)=\frac{47-20}{3}=9$$

So, the correct answer is **D**.

Solution 8.3

Reference: May 2001, Question 10, Answer B

Due to the symmetry of the region and the function, we have:

$$\overline{C(x,y)}=\frac{\iint_D C(x,y)\,dA}{\text{Area}(D)}=\frac{\iint_D C(x,y)\,dA}{16}$$

$$\iint_D C(x,y)\,dA=4\int_0^2\left(\int_0^2 22{,}500\left(8-x^2-y^2\right)dy\right)dx=90{,}000\int_0^2\left(\left(8-x^2\right)y-\frac{y^3}{3}\Bigg|_{y=0}^{2}\right)dx$$

$$=90{,}000\int_0^2\left(\frac{40}{3}-2x^2\right)dx=90{,}000\left(\frac{40x-2x^3}{3}\right)\Bigg|_0^2=1{,}920{,}000$$

$$\Rightarrow\overline{C(x,y)}=\frac{1{,}920{,}000}{16}=120{,}000$$

So, the correct answer is **B**.

Solution 8.4

Reference: November 2001, Question 2, Answer C

We have:

$$\iint_R \left(4f(x,y)-2\right)dA = 4\underbrace{\iint_R f(x,y)\,dA}_{\text{given as 6}} - 2\underbrace{\iint_R 1\,dA}_{\text{Area } R=2} = 4\times 6-2\times 2=20$$

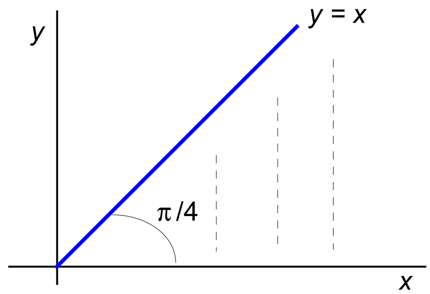
So, the correct answer is **C**.

Solution 8.5

Reference: Sample Exam, Question 6, Answer C

This is another question that will be easier to solve using polar coordinates.

Let's start by drawing a picture of the region of integration:



In polar coordinates, the region is described in polar coordinates by $0 \le \theta \le \pi/4$, $0 \le r < \infty$.

So, we have:

$$\int_{x=0}^{\infty}\left(\int_{0}^{x}\left(1+x^2+y^2\right)^{-2}dy\right)dx = \int_{\theta=0}^{\pi/4}\left(\int_{r=0}^{\infty}\left(1+r^2\right)^{-2} r\,dr\right)d\theta$$

$$= \int_{\theta=0}^{\pi/4}\left(\left(-\frac{1}{2(1+r^2)}\Bigg|_{r=0}^{\infty}\right)\right)d\theta$$

$$= \int_{\theta=0}^{\pi/4}\frac{1}{2}d\theta = \pi/8$$

So, the correct answer is **C**.