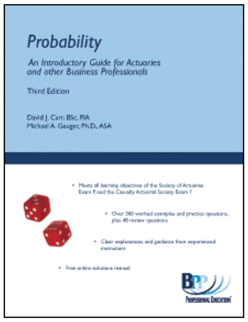


# Probability, Third Edition

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## Solutions to practice questions – Chapter 2

### Solution 2.1

(i) There are $n_1 = 4$ ways to choose the color of the cap, $n_2 = 6$ ways to choose the color of the shirt, and $n_3 = 2$ ways to choose the color of the pants. So the number of different color combinations for the uniform is:

$$n_1 n_2 n_3 = 4 \times 6 \times 2 = 48$$

(ii) There are $n_1 = 4$ ways to choose the color of the cap, $n_2 = 4$ ways to choose the color of the shirt, and $n_3 = 1$ way to choose the color of the pants. So the number of different color combinations for the uniform is:

$$n_1 n_2 n_3 = 4 \times 4 \times 1 = 16$$

(iii) First, note that the set of pants colors is a subset of the set of cap colors, which in turn is a subset of the shirt colors. If we first select the pant colors, there are $n_1 = 2$ choices. Once the color of the pants is determined, there are $n_2 = 4 - 1 = 3$ ways to choose a different color for the cap. Once the color of both the pants and the cap have been chosen, there are $n_3 = 6 - 2 = 4$ ways to choose a different color for the shirt. So the number of different color combinations for the uniform is:

$$n_1 n_2 n_3 = 2 \times 3 \times 4 = 24$$

### Solution 2.2

(i) The number of ordered samples of 3 objects from a set of 10 objects with replacement is:

$$10^3 = 1,000$$

(ii) The number of ordered samples of 3 objects from a set of 10 objects without replacement is:

$$_{10}P_3 = 10 \times 9 \times 8 = 720$$

(iii) This part of the problem is quite difficult. Let $n_1, n_2$, and $n_3$ be the number of possible choices for the first, second and third balls chosen. Then notice that the values of $n_2$ and $n_3$ depend on whether the previous balls were odd or even. The following table gives a breakdown of all the possible outcomes:

| Ball 1 | $n_1$ | Ball 2 | $n_2$ | $n_3$ | $n_1 n_2 n_3$ |
|---|---|---|---|---|---|
| Odd | 5 | Odd | 4 | 8 | 160 |
| Odd | 5 | Even | 5 | 9 | 225 |
| Even | 5 | Odd | 5 | 9 | 225 |
| Even | 5 | Even | 5 | 10 | 250 |

Hence the total number of possible outcomes if the odd balls are not replaced is:

$$160+225+225+250=860$$

## *Solution 2.3*

Any runner can win at most one medal. So we are choosing an ordered sample of 3 objects from a set of 30 objects without replacement. The number of possible outcomes is:

$${}_{30}P_3 = 30\times 29\times 28 = 24{,}360$$

## *Solution 2.4*

The number of heads could be 3, 4 or 5. Since the order of the outcomes does not matter, the number of possible outcomes that include at three or more heads is:

$${}_5C_3 + {}_5C_4 + {}_5C_5 = 10+5+1 = 16$$

## *Solution 2.5*

(i) The seven distinct letters in WYOMING can be rearranged in $7! = 5{,}040$ ways.

(ii) Since there are 2 A's in ARIZONA, the number of ordered arrangements is:

$$\frac{7!}{2!} = 2{,}520$$

(iii) There are 13 letters in the word MASSACHUSETTS. The letter A occurs twice, the letter T occurs twice, and the letter S occurs four times. Each of the other five letters occurs just once. So the number of ordered rearrangements is:

$$\frac{13!}{4!2!2!} = 64{,}864{,}800$$

## *Solution 2.6*

The number of ways to select three groups of six people is:

$$\frac{18!}{6!6!6!} = 17{,}153{,}136$$

 

## *Solution 2.7*

Let $E$ denote the event of 3 or more heads. It is much easier to count the ways in which we can observe 2 or fewer heads. The required probability is:

$$\Pr(E)=1-\Pr(E')=1-\Pr(H_0)-\Pr(H_1)-\Pr(H_2)$$

where $H_i$ denotes the event that $i$ heads are observed before the game stops.

It is easy to see that $\Pr(H_0)=0.5^3$ since a tail must occur on 3 consecutive independent tosses of a fair coin.

One possible way that $H_1$ can occur is the sequence $HTTT$. The probability of this sequence is $0.5^4$. Since the final toss must be a tail, there are ${}_3C_1=3$ ways to rearrange $HTT$, so we have:

$$\Pr(H_1)=3\times 0.5^4$$

One possible way that $H_2$ can occur is $HHTTT$. The probability of this sequence is $0.5^5$. Since the final toss must be a tail, there are ${}_4C_2=6$ ways to rearrange $HHTT$, so we have:

$$\Pr(H_2)=6\times 0.5^5$$

Finally, we have:

$$\begin{aligned}\Pr(E)&=1-\Pr(E')=1-\Pr(H_0)-\Pr(H_1)-\Pr(H_2)\\ &=1-\left(0.5^3+3\times 0.5^4+6\times 0.5^5\right)=0.50\end{aligned}$$

## *Solution 2.8*

The order in which the funds is selected does not matter. The 10 funds can be selected from the 100 funds in ${}_{100}C_{10}$ ways. The number of ways of selecting 3 funds from the top 25 funds is ${}_{25}C_3$, the number of ways of selecting 5 funds from the middle 50 funds is ${}_{50}C_5$, and the number of ways of selecting 2 of the bottom 25 funds is ${}_{25}C_2$. So the required probability is:

$$\frac{{}_{25}C_3\times{}_{50}C_5\times{}_{25}C_2}{{}_{100}C_{10}}=\frac{(2,300)(2,118,760)(300)}{1.731031\times 10^{13}}=0.0845$$

## *Solution 2.9*

(i) Each of the ${}_{48}C_6=12,271,512$ six-digit combinations is equally likely. There are ${}_9C_2=36$ ways to select two of the numbers 1-9, and there are ${}_{39}C_4=82,251$ ways to choose four of the numbers 10-48. So, the probability that exactly two of the winning six numbers are single digit numbers is:

$$\frac{({}_9C_2)({}_{39}C_4)}{{}_{48}C_6}=\frac{36\times 82,251}{12,271,512}=0.2413$$

(ii) Similarly, that probability that two of the winning numbers are single digit and the rest of the winning combination consists of one number from 10-19, one number from 20-29, one number from 30-39, and one number from 40-48 is:

$$\frac{({}_9C_2)({}_{10}C_1)({}_{10}C_1)({}_{10}C_1)({}_9C_1)}{{}_{48}C_6}=\frac{36\times 10\times 10\times 10\times 9}{12,271,512}=0.0264$$

 

## *Solution 2.10*

Let *M* denote the event that the largest claim for a week is on a medical policy. Similarly, let *A* and *H* respectively denote the event that the largest claim during a week is from an automobile policy or a homeowner's policy.

The event that the largest claim is from a medical policy on at least 2 more occasions than from an automobile policy corresponds to the following (unordered) combinations:

*MMHH* *MMMA* *MMMH* *MMMM*

The probability of the sequence *MMHH* is $0.6^2 \times 0.1^2$ and there are $4!/(2!2!)$ possible rearrangements of these 4 letters. Using the same idea with the other sequences listed above, we see that the probability that the largest claim is from a medical policy on at least 2 more occasions than from an automobile policy is:

$$\frac{4!}{2!2!}0.6^2 \times 0.1^2 + \frac{4!}{3!}0.6^3 \times 0.3 + \frac{4!}{3!}0.6^3 \times 0.1 + 0.6^4 = 0.4968$$

## *Solution 2.11*

Let *N* denote the event that no accidents occur in a month, and let *A* denote the complementary event that there is at least one accident in a month. The critical time period to consider is 7 months. At the end of 7 months, we will have witnessed either at least two months with accidents, or at least six accident-free months.

The probability that the first 7 months are all accident-free is:

$$0.76^7 = 0.1465$$

The probability that there are exactly 6 accident-free months in the first 7 months is:

$${}_7C_1 \times 0.76^6 \times 0.24 = 0.3237$$

Hence the required probability is $0.1465 + 0.3237 = 0.4702$.

*If you are attempting this question after studying Chapter 5, you can calculate this probability using the negative binomial distribution. You would calculate* $\Pr(X \ge 6) = 1 - \Pr(X < 6)$, *where* $X$ *has a negative binomial distribution with parameters* $p = 0.24$ *and* $r = 2$.

## *Solution 2.12*

From the description in the question, the event of interest is equivalent to one minor and one moderate accident in the month, or two minor accidents in the month. So the probability of this event is:

$${}_2C_1 \times \Pr(\text{minor}) \times \Pr(\text{moderate}) + (\Pr(\text{minor}))^2 = 2 \times 0.5 \times 0.4 + 0.5^2 = 0.65$$

 

## *Solution 2.13*

The unfortunate coach will be fired if the Cadavers win 0, 1, or 2 of the next 7 road games.

The probability is thus:

$$\underbrace{{}_7C_0 \times 0.65^7}_{\text{7 losses}} + \underbrace{{}_7C_1 \times 0.35^1 \times 0.65^6}_{\text{6 losses, 1 win}} + \underbrace{{}_7C_2 \times 0.35^2 \times 0.65^5}_{\text{5 losses, 2 wins}} = 0.5323$$

## *Solution 2.14*

If you win 4 or 5 matches of the next 5, you must automatically win consecutive matches. The probability of at least 4 wins is:

$$0.4^5 + {}_5C_1 \times 0.4^4 \times 0.6^1 = 0.08704$$

There are ${}_5C_3 = 10$ different ways that you could win 3 matches. Only one of these ways would not result in consecutive wins (*ie* WLWLW). So the probability of winning 3 matches and winning consecutive matches is:

$$\left({}_5C_3 - 1\right) \times 0.4^3 \times 0.6^2 = 0.20736$$

There are exactly 4 ways that you could win two consecutive matches and yet win only 2 of the 5 matches (WWLLL, LWWLL, LLWWL, LLLWW). The probability of this outcome is:

$$4 \times 0.4^2 \times 0.6^3 = 0.13824$$

Hence, the total probability is 0.43264.

## *Solution 2.15*

There are 60 seats and you are in seat 4B, so there are ${}_{59}C_2 = 1,711$ equally likely ways to fill the two adjacent seats with a passenger or an empty space. Let $E$ be the event that a member of the college soccer team will occupy at least one of the two adjacent seats. It is simpler to count the number of combinations corresponding to the complementary event.

For the complementary event, occupiers are chosen from the other 44 passengers and empty seat assignments (60 minus you minus the team):

$$\Pr(E) = 1 - \Pr(E') = 1 - \frac{{}_{44}C_2}{1,711} = 1 - \frac{946}{1,711} = 0.4471$$

## *Solution 2.16*

The other positions in George's group can be filled in ${}_{23}C_2 = 253$ ways (23 is 24 people minus George). The number of ways in which these positions can be filled by students other than George's friends is ${}_{19}C_2 = 171$ (19 is 24 minus George minus 4 friends). Hence the required probability is:

$$\frac{{}_{19}C_2}{{}_{23}C_2} = \frac{171}{253} = 0.6759$$

## *Solution 2.17*

Consider the following table where $n$ is the number of George's friends. The probability that George has at least one friend in his group if he has $n$ friends in the class is:

$$\Pr(\text{At least 1 friend in group}) = 1 - \frac{{}_{23-n}C_2}{{}_{23}C_2} = 1 - \frac{(23-n)(22-n)}{23\times 22} = \frac{45n-n^2}{506}$$

From the following table, it is clear that he must have 7 friends for there to be at least a 50% chance that he has at least one friend in his group.

| $n$ | 4 | 5 | 6 | 7 |
|---|---|---|---|---|
| $\Pr(\text{At least 1 friend in group})$ | 0.324 | 0.395 | 0.462 | 0.526 |

*Alternatively, using the quadratic formula, we could proceed as follows:*

$$\frac{45n-n^2}{506} = 0.5 \quad \Rightarrow \quad n^2 - 45n + 253 = 0 \quad \Rightarrow \quad n = 38.4 \text{ or } 6.6$$

*Since $n$ must be an integer less than 23, we have $n = 7$.*

## *Solution 2.18*

To not get a pair when the 4th card is drawn, the 4th card must not be a jack, a 2 or a 9. There are 49 cards in the pack before the 4th card is drawn and 40 of these are not jack, 2 or 9. So the probability of not completing a pair when the 4th card is drawn is $\frac{40}{49}$.

Now suppose we have 4 cards with no pairs (jack, 2, 9, X). To not get a pair when the 5th card is drawn, the 5th card must not be a jack, a 2, a 9 or an X. There are 48 cards in the pack before the 5th card is drawn and 36 of these are not jack, 2, 9 or X. So the probability that the 5th card does not complete a pair (given that there are no pairs so far) is $\frac{36}{48}$.

Hence:

$$\Pr(\text{no pairs}) = \frac{40}{49} \times \frac{36}{48} = 0.6122$$

and the probability of ending up with at least one pair is $1 - 0.6122 = 0.3878$.

*Alternatively, you could argue as follows. There are ${}_{49}P_2 = 49\times 48 = 2,352$ ways to choose the fourth and fifth cards in his hand. Once again it is simpler to compute the probability of the complementary event, namely, he will not end up with a pair.*

*The number of ways to choose a fourth card that is neither a jack, nor a two, nor a nine is ${}_{40}C_1 = 40$.*

*Once this card has been chosen, we must choose a fifth card that does not match any of the other four cards, and there are ${}_{36}C_1 = 36$ ways to do so.*

*So the probability that he will end up with at least one pair is:*

$$1 - \frac{40\times 36}{2,352} = 1 - \frac{1,440}{2,352} = 0.3878$$

 

## *Solution 2.19*

The number of ways to match exactly 3 of the 6 winning numbers is $\left({}_{6}C_{3}\right)\left({}_{43}C_{3}\right)=246,820$. In a similar fashion we can count the number of ways of matching exactly 4, 5, or 6 of the winning numbers. The probability of matching at least 3 of the 6 winning numbers is:

$$\frac{\left({}_{6}C_{3}\right)\left({}_{43}C_{3}\right)+\left({}_{6}C_{4}\right)\left({}_{43}C_{2}\right)+\left({}_{6}C_{5}\right)\left({}_{43}C_{1}\right)+\left({}_{6}C_{6}\right)\left({}_{43}C_{0}\right)}{{}_{49}C_{6}}=\frac{260,624}{13,983,816}=0.0186$$

## *Solution 2.20*

Let $H$ denote the event that a new driver is high risk, let $M$ denote a moderate risk driver, and let $L$ denote a low risk driver. Then we have:

$$\Pr(H)=0.2\ ,\ \Pr(M)=0.3\ ,\ \Pr(L)=0.5$$

For there to be at least 2 more high risk drivers than low risk drivers among the next 4 new drivers we could have any of the following:

$$2H,2M,0L \qquad \Pr=\frac{4!}{2!2!0!}\times 0.2^{2}\times 0.3^{2}=0.0216$$

$$3H,1M,0L \qquad \Pr=\frac{4!}{3!1!0!}\times 0.2^{3}\times 0.3^{1}=0.0096$$

$$3H,0M,1L \qquad \Pr=\frac{4!}{3!0!1!}\times 0.2^{3}\times 0.5^{1}=0.0160$$

$$4H,0M,0L \qquad \Pr=\frac{4!}{4!0!0!}\times 0.2^{4}=0.0016$$

Hence the total probability is:

$$0.0216+0.0096+0.0160+0.0016=0.0488$$